Lecture 16) Other electrodymics equations in Lorentz-covariant form

Two of Maxwell's equations can be written as:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$$

For $\nu = 0$

$$\partial_\mu F^{\mu 0} = 0 + \frac{\partial}{\partial x^1}\left(\frac{E_1}{c}\right) + \frac{\partial}{\partial x^2}\left(\frac{E_2}{c}\right) + \frac{\partial}{\partial x^3}\left(\frac{E_3}{c}\right)$$

$$= \nabla \cdot \left(\frac{\underline{\underline{E}}}{c}\right) = \mu_0 j^0 = \mu_0 c \rho$$

$$\therefore \nabla \cdot \underline{\underline{E}} = \frac{\rho}{\varepsilon_0}$$

Reminder

$$
\Lambda^{\delta}{}_{\gamma} = \begin{array}{c|c}
 & \gamma \longrightarrow \\
\delta \downarrow & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left(\begin{array}{cc|cc}
\gamma & -\gamma\beta & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\
-\gamma\beta & \gamma & & \\
\hline
\multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\
 & & 0 & 1
\end{array} \right)
\end{array}
$$

$$
F^{\alpha\beta} = \begin{array}{c|c}
 & \beta \longrightarrow \\
\alpha \downarrow & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix}
0 & -E_1/c & -E_2/c & -E_3/c \\
E_1/c & 0 & -B_3 & B_2 \\
E_2/c & B_3 & 0 & -B_1 \\
E_3/c & -B_2 & B_1 & 0
\end{pmatrix}
\end{array}
$$

For $\nu = 1$

$$\partial_\mu F^{\mu 1} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{-E_1}{c}\right) + 0 + \frac{\partial (B_3)}{\partial x^2} + \frac{\partial}{\partial x^3}\left(-B_2\right)$$

$$= \left[-\frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} + \nabla \times \underline{B} \right]_1 = \mu_0 j^1$$

Exercise: repeat for the x^2 and x^3 components.

$$\therefore \quad \nabla \times \underline{B} = \mu_0 \underline{j} + \varepsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}$$

This equation is closely related to the wave equation

$$\mu_0 j^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right)$$

$$= \partial_\mu \partial^\mu A^\nu - \partial^\nu \underbrace{\left(\partial_\mu A^\mu \right)}_{=0 \text{ in the Lorenz gauge.}}$$

$$= \Box^2 A^\nu$$

An Exercise for you

Show that

$$\boxed{\partial^\mu F^{\nu\lambda} + \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} = 0}$$

correspond to the two homogeneous Maxwell's Equations

(Note the cyclic permutation of indices.)

Hint: try using $\left.\begin{array}{l} 1, 2, 3 \\ \text{and also } 0, 1, 2 \end{array}\right\}$ for μ, ν, λ

The Interaction of charged particles with the fields

$$\boxed{\frac{dp^\mu}{d\tau} = q\, F^{\mu\nu}\, u_\nu}$$

proper time maintains
left hand side as
a 4-vector.

right hand side is also a 4-vector.

Let's start with $\mu = 0$:

$$qF^{0\nu}u_{\nu} = q\left[0 + \left(-\frac{E_1}{c}\right)\left(-\gamma u_1\right) + \left(-\frac{E_2}{c}\right)\left(-\gamma u_2\right) + \left(-\frac{E_3}{c}\right)\left(-\gamma u_3\right)\right]$$

$$= q\gamma\,\frac{\underline{u}\cdot\underline{E}}{c}$$

For $\mu = 1$:

$$qF^{1\nu}u_{\nu} = q\left[\frac{E_1}{c}\gamma c + 0 + \left(-B_3\right)\left(-\gamma u_2\right) + B_2\left(-\gamma u_3\right)\right]$$

$$= q\gamma\left[\underline{E} + \underline{u}\times\underline{B}\right]_1$$

Exercise: repeat for the x^2 and x^3 components

$$\therefore\quad qF^{\mu\nu}u_{\nu} = q\gamma\left(\frac{\underline{u}\cdot\underline{E}}{c},\ \underline{E} + \underline{u}\times\underline{B}\right)$$

Exercise: obtain the same result by multiplying matrices $F^{\mu\nu}u_{\nu}$.

Write $dt = \gamma d\tau$ and consider the 0^{th} component

$$\frac{dp^0}{d\tau} = \gamma \frac{dp^0}{dt} = \gamma \frac{d}{dt}\left(\frac{\overbrace{\mathcal{E}}^{\text{Energy of particle}}}{c}\right) = q\gamma \frac{\underline{u}\cdot\underline{E}}{c}$$

$$\therefore \quad \frac{d\mathcal{E}}{dt} = \underbrace{q\,\underline{u}\cdot\underline{E}}$$

which is the rate of work on the charge q done by the E field

(N.B. B field does no work)

The Interaction of charged particles with the fields

$$\boxed{\frac{dp^\mu}{d\tau} = q\, F^{\mu\nu}\, u_\nu}$$

proper time maintains left hand side as a 4-vector.

right hand side is also a 4-vector.

Exercise for you

Show that

$$\frac{d\underline{p}}{dt} = q\left(\underline{E} + \underline{u} \times \underline{B}\right)$$ the Lorentz force law

can be obtained from the $\mu = 1, 2, 3$ components of our 4-vector $\frac{dp^\mu}{d\tau}$.