

For $\nu = 2$

$$\partial_\mu F^{\mu 2} = \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{E_2}{c} \right) + \frac{\partial}{\partial x^1} (-B_3) + 0 + \frac{\partial}{\partial x^3} (B_1)$$

$$= \left[-\frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} + \nabla \times \underline{B} \right]_2 = \mu_0 j^2$$

For $\nu = 3$

$$\partial_\mu F^{\mu 3} = \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{E_3}{c} \right) + \frac{\partial}{\partial x^1} (B_2) + \frac{\partial}{\partial x^2} (-B_1)$$

$$= \left[-\frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} + \nabla \times \underline{B} \right]_3 = \mu_0 j^3$$

as required.

For $\mu, \nu, \lambda = 1, 2, 3$

$$0 = \partial^1 F^{23} + \partial^3 F^{12} + \partial^2 F^{31}$$

$$= - \left[\frac{\partial}{\partial x^1} (-B_1) + \frac{\partial}{\partial x^3} (-B_3) + \frac{\partial}{\partial x^2} (-B_2) \right] = \nabla \cdot \underline{B}$$

↑
from $\partial^i = \frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x^i}$

$$\therefore \nabla \cdot \underline{B} = 0$$

For $\mu, \nu, \lambda = 0, 1, 2$

$$0 = \partial^0 F^{12} + \partial^2 F^{01} + \partial^1 F^{20}$$

$$= \frac{1}{c} \frac{\partial}{\partial t} (-B_3) + \left(-\frac{\partial}{\partial x^2}\right) \left(-\frac{E_1}{c}\right) + \left(-\frac{\partial}{\partial x^1}\right) \left(\frac{E_2}{c}\right)$$

$$= -\frac{1}{c} \left[\frac{\partial \underline{B}}{\partial t} + \nabla \times \underline{E} \right]_3$$

Similar results can be obtained for

$$\mu, \nu, \lambda = 0, 2, 3 \quad \text{and} \quad 0, 3, 1,$$

thus demonstrating that

$$\frac{\partial \underline{B}}{\partial t} + \nabla \times \underline{E} = 0$$

More generally, we could write

$$0 = \partial^0 F^{ij} + \partial^j F^{0i} + \partial^i F^{j0}$$

$$= \frac{1}{c} \frac{\partial}{\partial t} \left(-\epsilon_{ijk} B_k \right) + \left(-\frac{\partial}{\partial x^j} \right) \left(-\frac{E_i}{c} \right) + \left(-\frac{\partial}{\partial x^i} \right) \left(\frac{E_j}{c} \right)$$

$$= -\frac{\epsilon_{ijk}}{c} \frac{\partial B_k}{\partial t} - \frac{\epsilon_{ijk}}{c} [\nabla \times \underline{E}]_k$$

$$= -\frac{\epsilon_{ijk}}{c} \left[\frac{\partial B_k}{\partial t} + \nabla \times \underline{E} \right]_k$$

which shows that $\left[\frac{\partial \underline{B}}{\partial t} + \nabla \times \underline{E} \right]_k = 0 \quad \forall k$

Interaction of charged particles with the fields

Lets consider the rate of change of 4-momentum with proper time

$$\boxed{\frac{dp^\mu}{d\tau} = q F^{\mu\nu} u_\nu}$$

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Pretty much has to depend on these 4-vectors

$$= q \begin{bmatrix} 0 & -\frac{E_1}{c} & -\frac{E_2}{c} & -\frac{E_3}{c} \\ \frac{E_1}{c} & 0 & -B_3 & B_2 \\ \frac{E_2}{c} & B_3 & 0 & -B_1 \\ \frac{E_3}{c} & -B_2 & B_1 & 0 \end{bmatrix} \begin{bmatrix} \gamma c \\ -\gamma u_1 \\ -\gamma u_2 \\ -\gamma u_3 \end{bmatrix} = q \gamma \begin{bmatrix} \frac{E_1 u_1}{c} + \frac{E_2 u_2}{c} + \frac{E_3 u_3}{c} \\ E_1 + B_3 u_2 - B_2 u_3 \\ E_2 - B_3 u_1 + B_1 u_3 \\ E_3 + B_2 u_1 - B_1 u_3 \end{bmatrix}$$

$$= q \gamma \left(\frac{\underline{u} \cdot \underline{E}}{c}, \underline{E} + \underline{u} \times \underline{B} \right)$$

N.B. u_i here are components of 3D vector $\underline{u}$. No distinction between upper or lower indices

For $\mu=2$

$$\begin{aligned} q F^{2\nu} u_\nu &= q \left[\frac{E_2}{c} (\gamma c) + B_3 (-\gamma u_1) + 0 + (-B_1) (-\gamma u_3) \right] \\ &= \gamma q \left[\underline{E} + \underline{u} \times \underline{B} \right]_2 \end{aligned}$$

For $\mu=3$

$$\begin{aligned} q F^{3\nu} &= q \left[\frac{E_3}{c} (\gamma c) + (-B_2) (-\gamma u_1) + B_1 (-\gamma u_2) + 0 \right] \\ &= \gamma q \left[\underline{E} + \underline{u} \times \underline{B} \right]_3 \end{aligned}$$

General result for $\mu \neq 0$ or $\mu = i$, $i = 1, 2, 3$

$$\frac{dp^i}{dz} = \gamma \frac{dp^i}{dt} = q F^{iv} u_v$$

$$= q \left\{ F^{i0} u_0 + \epsilon_{ijk} (-B_k) (-\gamma u_j) \right\}$$

$$= q \left\{ \frac{E_i}{c} (\gamma c) + \gamma [\underline{u} \times \underline{B}]_i \right\}$$

$$= \gamma q [\underline{E} + \underline{u} \times \underline{B}]_i \quad \forall i$$

Here $v \neq 0$

$v = j \neq i$

u_j is component of
3D vector $\underline{u}$

$$\therefore \frac{d\underline{p}}{dt} = q \left(\underline{E} + \underline{u} \times \underline{B} \right) \quad \text{as required}$$