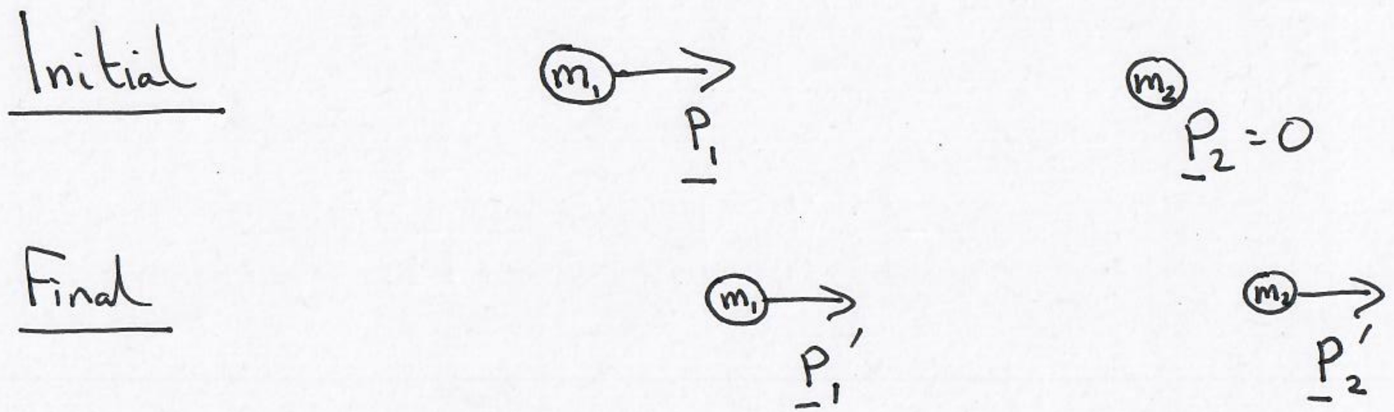
Lecture 17) Conservation Laws : "Global" vs. "Local"

For example, let's consider two positively charged particles



"Newtonian" view of "action at a distance" allows the transfer of momentum from m_1 to m_2 as long as

$$\Delta \underline{p}_1 = - \Delta \underline{p}_2 \quad \text{at each instant in time}$$

This is an example of a "global conservation law".

In special relativity only events that occur at the <u>same point in space</u> can unambiguously be called <u>simultaneous</u>

$\Rightarrow$ We require a "<u>local</u>" conservation law at each point in space time.

N.B. "local" conservation is a stronger requirement than "global"
(Avoid confusion with everyday language where "global" sounds more impressive than "local" ;-)

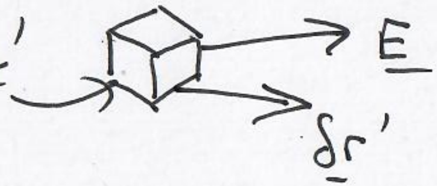
We have already written down one local conservation law in this course

$$\partial_\mu j^\mu = 0 \qquad \Rightarrow \text{ local conservation of charge.}$$

Local conservation of energy in electrodynamics

Consider an infinitessimal volume $\delta\tau'$ that is displaced by $\underline{\delta r}'$ in a field $\underline{E}$

$q = \rho\,\delta\tau'$



Energy gain by charges

$$\delta W = \rho\,\delta\tau'\,\underline{\delta r}'\cdot\underline{E}$$

$\therefore$ Rate of change of energy density of charges (= work done by E field)

$$\underline{E}\cdot\underline{v}\,\rho = \underline{E}\cdot\underline{j} = \underline{E}\cdot\left[\frac{1}{\mu_0}(\nabla\times\underline{B}) - \varepsilon_0\left(\frac{\partial\underline{E}}{\partial t}\right)\right]$$ [using Maxwell's equatio

$$-\varepsilon_0 \underline{E} \cdot \frac{\partial \underline{E}}{\partial t} = -\frac{\varepsilon_0}{2} \frac{\partial}{\partial t}\left(E^2\right)$$

Since $\nabla \cdot \left(\underline{E} \times \underline{B}\right) = \underline{B} \cdot \left(\nabla \times \underline{E}\right) - \underline{E} \cdot \left(\nabla \times \underline{B}\right)$ [vector identity (6)]

$$\underline{E} \cdot \left(\nabla \times \underline{B}\right) = -\nabla \cdot \left(\underline{E} \times \underline{B}\right) + \underline{B} \cdot \left(-\frac{\partial \underline{B}}{\partial t}\right)$$ [using Maxwell's equation]

$$= -\nabla \cdot \left(\underline{E} \times \underline{B}\right) - \frac{1}{2} \frac{\partial}{\partial t}\left(B^2\right)$$

$$\therefore \underline{E} \cdot \underline{j} = -\nabla \cdot \left(\frac{1}{\mu_0} \underline{E} \times \underline{B}\right) - \frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2\right)$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

$\boxed{(6) \quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})}$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

$$\therefore \quad \underline{E} \cdot \underline{j} + \nabla \cdot \underline{S} + \frac{\partial u}{\partial t} = 0$$

where $\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$ energy flux density (the Poynting Vector)

and $u = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$ energy density in the fields.

In words we can express this equation for the local conservation of energy in electrodynamics as

"Rate at which E field does work" + "flux of energy out of the surface" + "rate of change of field energy" = 0

Suggestion for further reading

Roger Barlow   Eur. J. Phys 11 (1990) 45.

talks about the physical consequences of local gauge invariance $A^\mu \Rightarrow A^\mu - \partial^\mu \chi$

(paper linked from the course website).