Lectures 19) Accelerating point charge – Vector calculus approach

In Lectures 14) & 15) we learned that for a point charge moving with constant velocity, A^μ, $\underline{E}$, $\underline{B}$ are centred at the "current" position of the charge.

In Lecture 18) we saw that when dealing with an accelerating point charge we need to look at what is going on at the "retarded" time, $t_{ret} = t - R/c$.

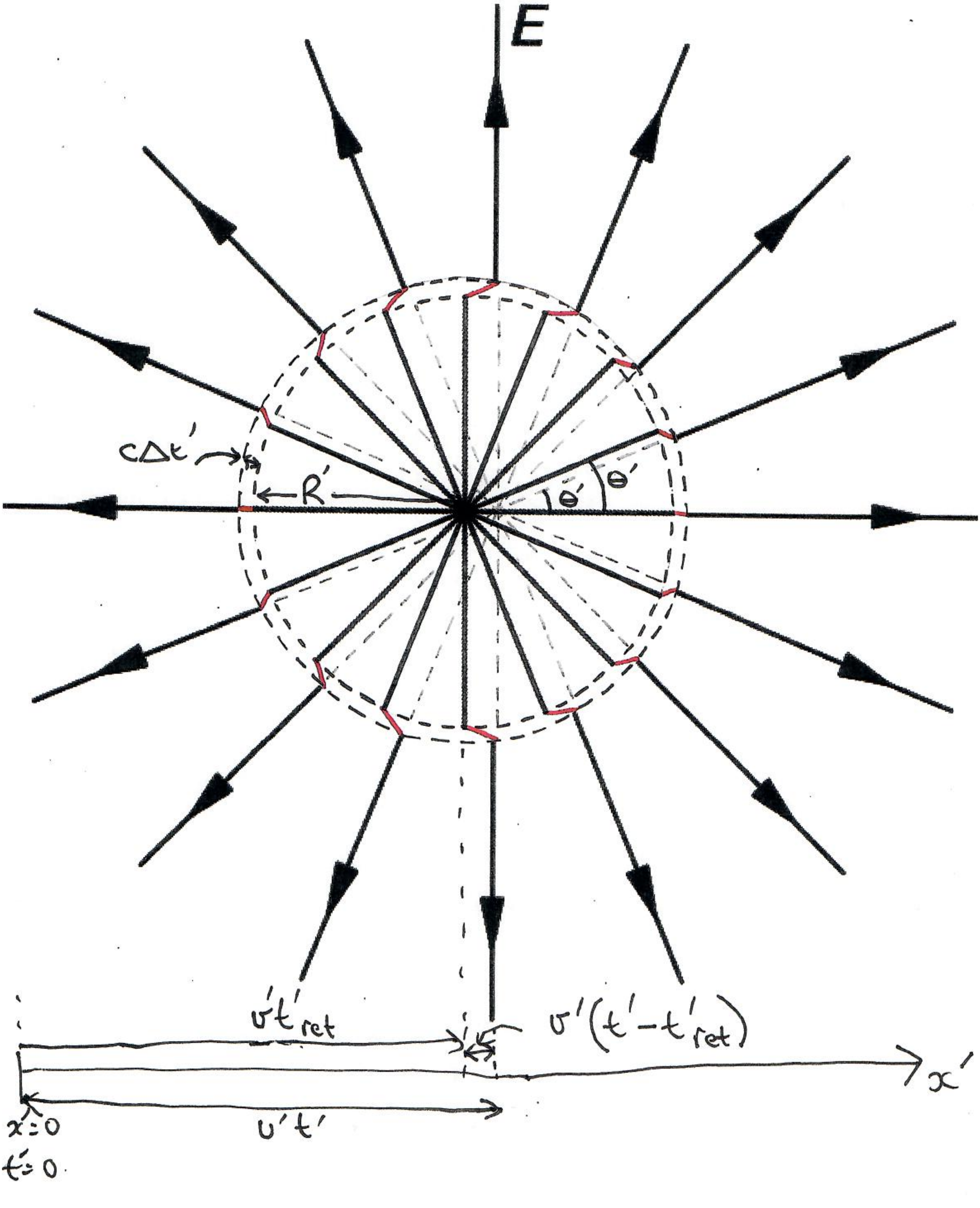
The plan of action

Lecture 19(a) Draw a diagram!

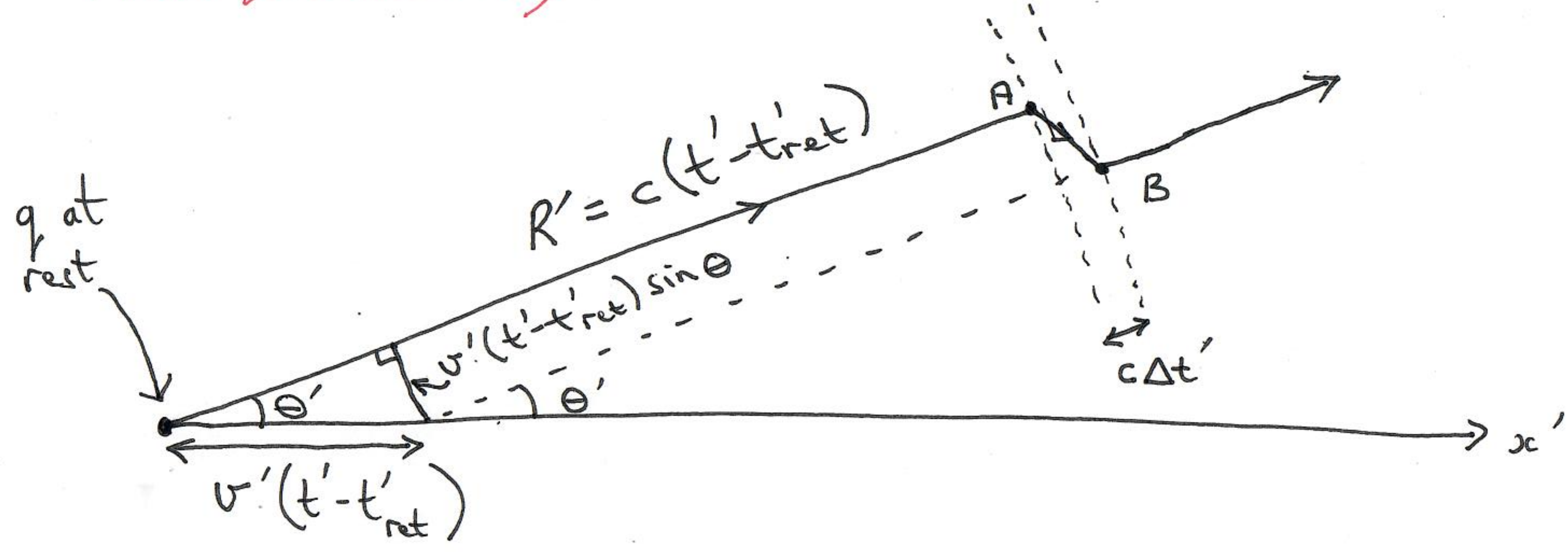
Re-write A^μ for moving point charge in terms of R_{ret}

Lecture 19(b) Use vector calculus methods to work out $\underline{E}$, $\underline{B}$ when $\dot{\underline{\beta}} = 0$

From Lecture 18)



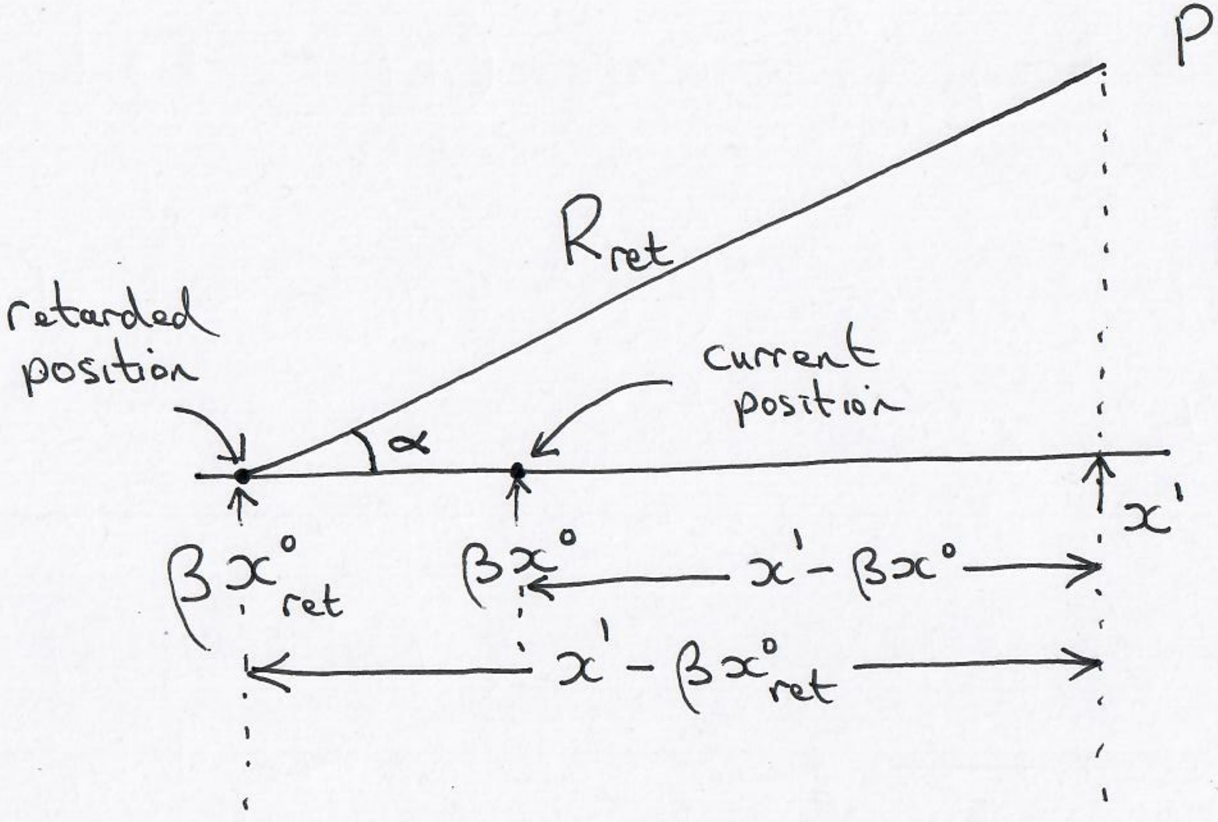
From Lecture 18)



Radial field at A: $E'_R = \frac{1}{4\pi\varepsilon_0}\frac{q}{(R')^2}$

Tangential component E'_T given by $\frac{E'_T}{E'_R} = \frac{v'(t'-t'_{ret})\sin\theta}{c\Delta t'}$

$$\therefore \boxed{E'_T = \frac{q}{4\pi\varepsilon_0}\frac{c(t'-t'_{ret})}{(R')^2 c}\frac{v'}{c\Delta t'}\sin\theta' = \frac{q}{4\pi\varepsilon_0 c}\frac{\dot{\beta}'}{R'}\sin\theta'}$$



$$(\;\;) = \left\{(x^2)^2 + (x^3)^2\right\}^{1/2}$$

From Lecture 14)

$$A^0 = \frac{q}{4\pi\varepsilon_0 c}\,\gamma\,\frac{1}{\left[\left(\gamma(x^1 - \beta x^0)\right)^2 + (\;\;)^2\right]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0 c}\,\frac{1}{\left\{\left[x - \beta x^0\right]^2 + (1-\beta^2)(\;\;)^2\right\}^{1/2}} \qquad (19.1)$$

Our aim is to express

$$\left\{ [x^1 - \beta x^0]^2 + (1-\beta^2)(\ )^2 \right\}$$

in terms of R_{ret} and α.

Some Usful Relations

$$(\ )^2 = R^2_{ret} \sin^2 \alpha \qquad (19.2)$$

$$x^1 - \beta x^0_{ret} = R_{ret} \cos \alpha \qquad (19.3)$$

From $t_{ret} = t - \frac{R_{ret}}{c}$

$$x^0 = x^0_{ret} + R_{ret} \qquad (19.4)$$

Using (19.4)

$$x^1 - \beta x^0 = x^1 - \beta\left(R_{ret} + x^0_{ret}\right)$$

using (19.3)

$$= x^1 - \beta x^0_{ret} - \beta R_{ret} = R_{ret}(\cos\alpha - \beta) \qquad (19.5)$$

Using (19.2) and (19.5) we can express

$$\left\{ [x^1 - \beta x^0]^2 + (1-\beta)^2 (\ )^2 \right\}$$

$$= R_{ret}^2 \left(\cos^2\alpha - 2\beta\cos\alpha + \beta^2 + (1-\beta^2)\sin^2\alpha \right)$$

$$= R_{ret}^2 \left(\cos^2\alpha + \sin^2\alpha - 2\beta\cos\alpha + \beta^2(1-\sin^2\alpha) \right)$$

$$= R_{ret}^2 \left(1 - 2\beta\cos\alpha + \beta^2\cos^2\alpha \right) = \left[R_{ret}(1 - \beta\cos\alpha) \right]^2$$

$$= \left[R_{ret} \left(1 - \underline{\beta} \cdot \hat{\underline{R}}_{ret} \right) \right]^2 \qquad (19.6)$$

Substitute (19.6) into (19.1)

$$A^0 = \frac{q}{4\pi\varepsilon_0 c} \frac{1}{\left[R(1 - \underline{\beta} \cdot \hat{\underline{R}}) \right]_{ret}} \qquad (19.7)$$

where $[\ ]_{ret}$ indicates that $\underline{R}$, $\underline{\beta}$ are evaluated at the retarded time.

Potentials in frame S

From Lecture 14)

L.T. $\frac{V}{c} = A^0 = \gamma(A'^0 + \beta A'^1) = \gamma A'^0$

(Note: inverse Lorentz Transformation.)

$$\therefore \; A^0 = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{1}{\left[\left(\gamma[x^1-\beta x^0]\right)^2+(x^2)^2+(x^3)^2\right]^{1/2}}$$

$$A^1 = \gamma(\beta A'^0 + A'^1) = \gamma\beta A'^0 = \beta A^0$$

$$= \frac{q}{4\pi\varepsilon_0 c}\gamma\beta\frac{1}{\left[\left(\gamma[x^1-\beta x^0]\right)^2+(x^2)^2+(x^3)^2\right]^{1/2}}$$

$$A^2 = A^3 = 0$$

$\underline{A} = \underline{\beta} A^0$

Exercise for you: Cross-check that $A'^{\mu}A'_{\mu} = A^{\mu}A_{\mu}$.

From Lecture 14) we can write

$$\underline{A} = \underline{\beta} A^0$$

and hence

$$\underline{A} = \frac{q}{4\pi\varepsilon_0 c}\left[\frac{\underline{\beta}}{R(1-\underline{\beta}\cdot\hat{\underline{R}})}\right]_{ret} = \left[\underline{\beta}\right]_{ret} A^0 \qquad (19.8)$$

(19.7) and (19.8) are called the "<u>Liénard-Wiechert</u>" potentials for a point charge.