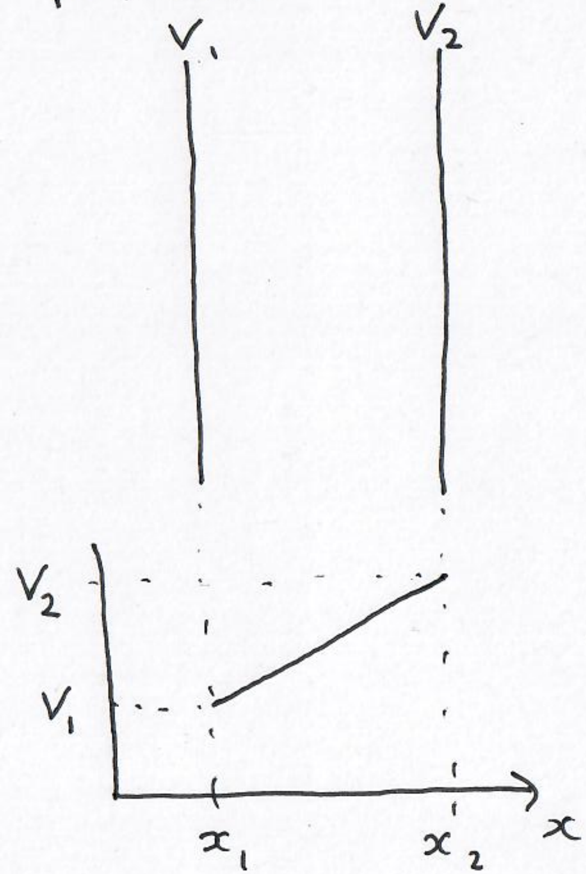
Lecture 3) Solutions to Laplace's Equation: "Boundary Value Problems"

In 1-Dimension

Infinite parallel-plate conductors held at constant potential



$$\frac{d^2V}{dx^2} = 0$$

"Trivial solution" $\quad V(x) = ax + b$

Illustrates two important properties of solutions to Laplace's equation that are also valid in 2D and 3D.

(A) $V(x) = \frac{1}{2}\left[V(x+c) + V(x-c)\right]$

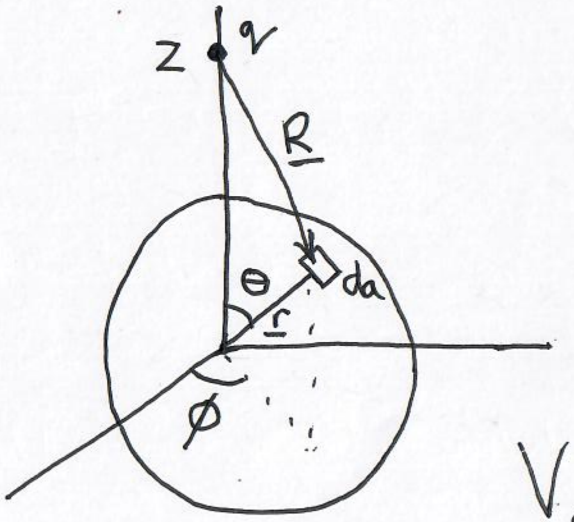
N.B. c does not have to be infinitessimally small.

$\Rightarrow$ (B) There can be no local maxima or minima in V.

Proof of (A) in 3D

Consider V due to a point charge q.

Without loss of generality we can find V at the origin for a charge q located at $\underline{r}' = z\,\hat{\underline{z}}$.



Evaluate $V_{av.}$ over spherical surface of radius $r < z$ centred at the origin

N.B. r is not infinitesimally small.

$$V_{av.} = \frac{1}{4\pi r^2} \oint V(\underline{r})\, da = \frac{1}{4\pi r^2} \frac{q}{4\pi \varepsilon_0} \oint \frac{da}{R} \quad \text{(Eqn 3.1)}$$

Exercise for you:

Starting from Eqn 3.1

using $da = r^2 \sin\theta \, d\theta \, d\phi$

and $R = \left[z^2 + r^2 - 2zr\cos\theta\right]^{1/2}$

show that $\oint \frac{da}{R} = \frac{4\pi r^2}{z}$ for all $r < z$.

Subsituting result back into (3.1) gives

$$V_{av} = \frac{q}{4\pi\varepsilon_0}\frac{1}{z}$$ as expected for V at distance z from charge q.

Therefore also in 3-D where $\nabla^2 V = 0$

(A) V is given by average over symmetrically located neighboring points.

$\Rightarrow$ (B) There can be no local minima / maxima

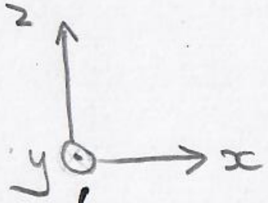
E.g., If there were to be a local minimum, V_0, all points on surrounding surface would have $V > V_0$, violating (A)

$\Longrightarrow$ <u>Earnshaw's Theorem.</u>

A charged particle cannot be held in a position of stable equilibrium by electrostatic forces alone.

N.B. Saddle points are allowed!

(but peaks are not)



$$\frac{\partial^2 z}{\partial x^2} > 0$$

$$\frac{\partial^2 z}{\partial y^2} < 0$$

can satisfy

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

A Uniqueness Theorem

The solution to Laplace's equation in a volume V is uniquely determined if on the boundary surface S we specify either

(a) V (Dirichlet)

or (b) $\nabla V \cdot \hat{\underline{n}}$ where $\hat{\underline{n}}$ is normal to surface S (Neumann)

Justification

Suppose there were two different solutions satisfying

$\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$ within this volume

and satisfying $V_1 = V_2$ on the boundary surface.

Consider $f = V_1 - V_2$: This satisfies Laplace's equation

$$\nabla^2 f = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

On the boundary $V_1 = V_2$ so $f = 0$.

$\therefore$ Since there can be no maxima or minima within the volume $f = 0$ everywhere

$\therefore$ $V_1 = V_2$ the solution is unique.

$\Longrightarrow$ If you find a solution to $\nabla^2 V = 0$ by <u>any</u> method that satifies the boundary conditions ...

it has to be <u>the</u> solution.

ugly
beautiful or
guesswork!