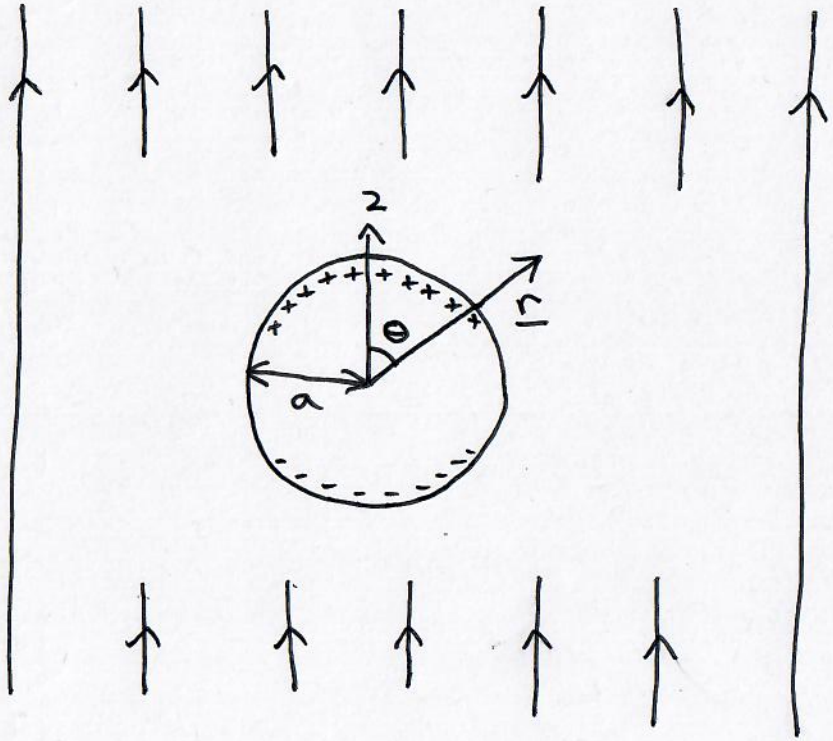
Lecture 4) Example of Laplace's Equation in Spherical Polar Coordinates

$\underline{E} = E_0 \hat{\underline{z}}$



- Potential inside sphere = 0
- E inside sphere = 0

What happens?

- Charge flows
  +ve charge at the top of the sphere
  −ve charge at the bottom of the sphere

$\Rightarrow$ Surface of sphere is an equipotential

Let's choose : $V = 0$ at surface of sphere
Centre of sphere at the origin

- At surface of sphere field must be radial

$$\underline{E}(r, \theta, \phi) = \underline{E}(a, \theta)\, \hat{\underline{r}}$$

ignore because of symmetry in ϕ

- Large distances from the sphere $r \to \infty$

$V = -E_0 z = -E_0 r \cos\theta$ (in our spherical polar coordinates)

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

General Formalism for 2D (r, θ) systems (because of symmetry in ϕ)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \qquad \text{(Eqn 4.1)}$$

Employ separation of variables

$$V(r, \theta) = R(r)\,\Theta(\theta) \qquad \text{(Eqn 4.2)}$$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{l(l+1)} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{-l(l+1)} = 0$$

constants $l(l+1)$ $\quad -l(l+1)$

General solution for equation in R

$$R(r) = Ar^{l} + \frac{B}{r^{l+1}}$$

Exercise: verify that this expression is a solution to the equation in R.

General solution for equation in Θ

$$\Theta(\theta) = P_l(\cos\theta)$$

Exercise: verify that P_0, P_1, P_2 ... are solutions to the equation in Θ

where $P_l(\cos\theta)$ are the Legendre Polynomials given by:

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = (3\cos^2\theta - 1)/2$$

$$P_l(\cos\theta) = \frac{1}{2^l l!}\left(\frac{d}{d(\cos\theta)}\right)^l (\cos^2\theta - 1)^l$$

General solution for $V(r, \theta)$

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$\int_0^{\pi} V \, P_l(\cos\theta) \sin\theta \, d\theta = \frac{2}{2l+1} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right)$$

However if V can be expressed as some function $f(\cos\theta)$ it can be simpler to obtain A_l and B_l "by inspection".

Return to our specific problem and apply the boundary conditions

- $r \to \infty$, $V = -E_0 r \cos\theta$

By comparison with the general solution we can see that

$A_l = 0$ except for the $l = 1$ term :

$$\therefore \quad A_1 r \cos\theta = -E_0 r \cos\theta$$

$$V = -E_0 r \cos\theta + \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

- $r = a$ (surface of sphere), $V = 0$

True for all θ only if $B_l = 0$ except for the $l = 1$ term

$$0 = -E_0 a \cos\theta + \frac{B_1}{a^2} \cos\theta \quad \therefore \quad B_1 = E_0 a^3$$

$$\therefore \quad V(r,\theta) = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

External field   Induced charge on sphere

Electric Field

$$\underline{E} = -\nabla V = -\left(\frac{\partial V}{\partial r} \hat{\underline{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\underline{\theta}} \right)$$

$$= E_0 \left(1 + \frac{2a^3}{r^3} \right) \cos\theta \, \hat{\underline{r}} - E_0 \left(1 - \frac{a^3}{r^3} \right) \sin\theta \, \hat{\underline{\theta}}$$

At $r = a$:

$\underline{E}$ $\quad E_\theta = 0$

$$E_r = 3 E_0 \cos\theta = \underbrace{\frac{\sigma(\theta)}{\varepsilon_0}}_{\text{from Gauss's theorem}}$$

(since $E = 0$ inside the sphere)

$$\therefore \sigma(\theta) = 3 \varepsilon_0 E_0 \cos\theta$$

Note

For systems without symmetry in ϕ solutions in terms of "Spherical Harmonics" $Y_l^m(\theta, \phi)$

"Extra curricular" reading see, e.g.,

Heald and Marion : Section 3.4

Jackson : Chapter 3.5