Verify general solution in $\mathbb{R}$

$$\frac{d}{dr}\left(r^2 \frac{d\mathbb{R}}{dr}\right) = l(l+1)\,\mathbb{R}$$ is the differential equation in $\mathbb{R}$

$$\frac{d}{dr}\left(r^2 \frac{d}{dr}\left[A r^l + \frac{B}{r^{l+1}}\right]\right) = \frac{d}{dr}\left(r^2\left[l A r^{l-1} - (l+1)\frac{B}{r^{l+2}}\right]\right)$$

$$= \frac{d}{dr}\left(l A r^{l+1} - (l+1)\frac{B}{r^l}\right) = l(l+1) A r^l + l(l+1)\frac{B}{r^{l+1}}$$

$$= l(l+1)\,\mathbb{R}$$, as required.

Verifying solutions to the differential equation for Θ

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta$$ is the relevant equation

For $P_0 = 1$

$$\frac{d}{d\theta}\left(\sin\theta \frac{d[1]}{d\theta}\right) = 0$$, as required for $l = 0$

For $P_1 = \cos\theta$

$$\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}[\cos\theta]\right) = -\frac{d}{d\theta}\left(\sin^2\theta\right) = -2\sin\theta\cos\theta$$

$$= -1(1+1)\sin\theta\,P_1$$, as required for $l = 1$

For $P_2 = (3\cos^2\theta - 1)/2$

$$\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\left[\frac{3\cos^2\theta - 1}{2}\right]\right) = \frac{d}{d\theta}\left(\sin\theta\left[-3\cos\theta\sin\theta\right]\right)$$

$$= -3\frac{d}{d\theta}\left[\sin^2\theta\cos\theta\right] = -3\left[2\sin\theta\cos^2\theta - \sin^3\theta\right]$$

$$= -3\sin\theta\left(3\cos^2\theta - 1\right) \qquad \left\{ \text{since } \sin^3\theta = \sin\theta(1-\cos^2\theta) \right.$$

$$= -2(2+1)\sin\theta\, P_2 \quad , \quad \text{as required for } l = 2$$