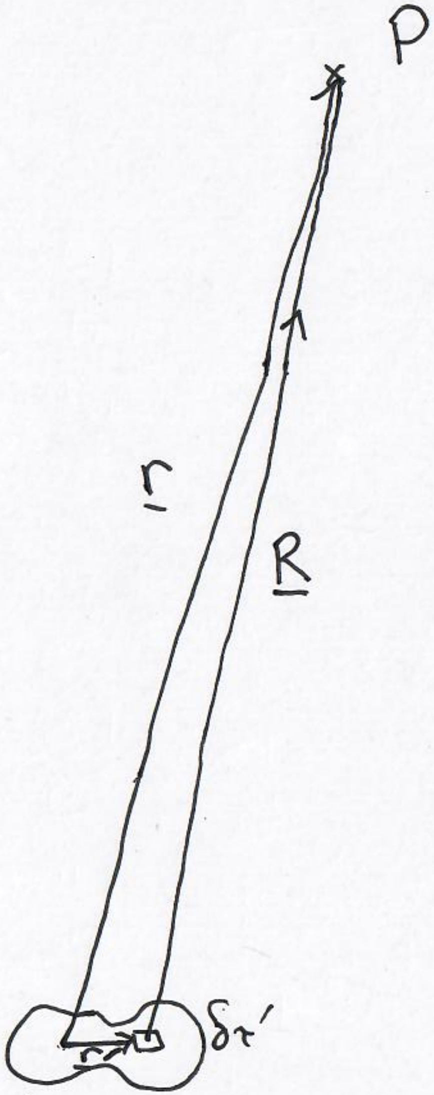
Lecture 5) Multipole Expansions in Electrostatics



$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\underline{r}')}{R} \, d\tau'$$

where $\underline{R} = \underline{r} - \underline{r}'$

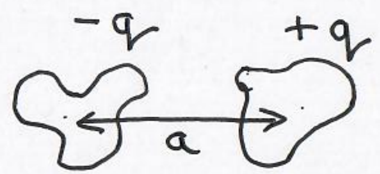
Sometimes it is useful to consider approximate expressions for $V(\underline{r})$ for the case $r \gg r'$

Nett charge $Q = \int \rho(r') \, d\tau'$

1st approximation: $V_1 \approx \frac{1}{4\pi\varepsilon_0} \frac{Q}{r}$ for $\underline{R} \approx \underline{r}$

"monopole" approximation

What if $Q = 0$ overall, but positive and negative charge distributions do not perfectly cancel one another?



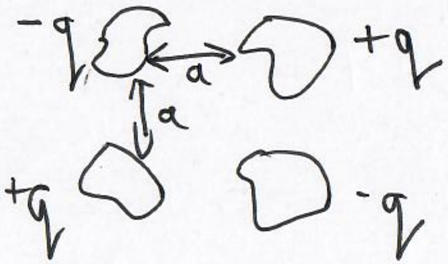
$r \gg a$

Can we guess the functional dependence of V on a and r

Seems natural to assume that $V \propto \dfrac{a}{r^2}$

Guess that $$V = \frac{1}{4\pi\varepsilon_0}\frac{q\,a}{r^2}$$

"dipole" term.



$$V \propto \frac{a^2}{r^3}$$

"quadrupole" term.

Please read Griffiths Introduction to Electrodynamics page 152-153.

$$\frac{1}{\imath} \Leftrightarrow \frac{1}{R}$$

Insert series expansion of $\frac{1}{R}$ in terms of powers of $\frac{1}{r}$ into the expression for $V(\underline{r})$ to obtain

$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int_v (r')^n P_n(\cos\alpha)\, \rho(\underline{r}')\, d\tau'$$

N.B. r is a constant as far as the integration is concerned.

Writing out the first few terms, which may be sufficient for $r \gg r'$

$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\left\{ \frac{1}{r}\int \rho(\underline{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\underline{r}')\,d\tau' \right.$$

"monopole"

$$\left. + \frac{1}{r^3}\int (r')^2\left[\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right]\rho(\underline{r}')\,d\tau' + \ldots \right.$$

This is called the Multipole expansion for $V(\underline{r})$

Dipole term

$$r' \cos\alpha = \hat{\underline{r}} \cdot \underline{r}'$$

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{\underline{r}} \cdot \int \underline{r}' \rho(\underline{r}') \, d\tau' = \frac{1}{4\pi\varepsilon_0} \frac{\hat{\underline{r}} \cdot \underline{p}}{r^2}$$

where $\underline{p} = \int \underline{r}' \rho(\underline{r}') \, d\tau'$ is called the "dipole moment"

It depends only on the distribution of charge and not $\underline{r}$!

Consider a point charge q at a position $\underline{a}$ $(a << r)$

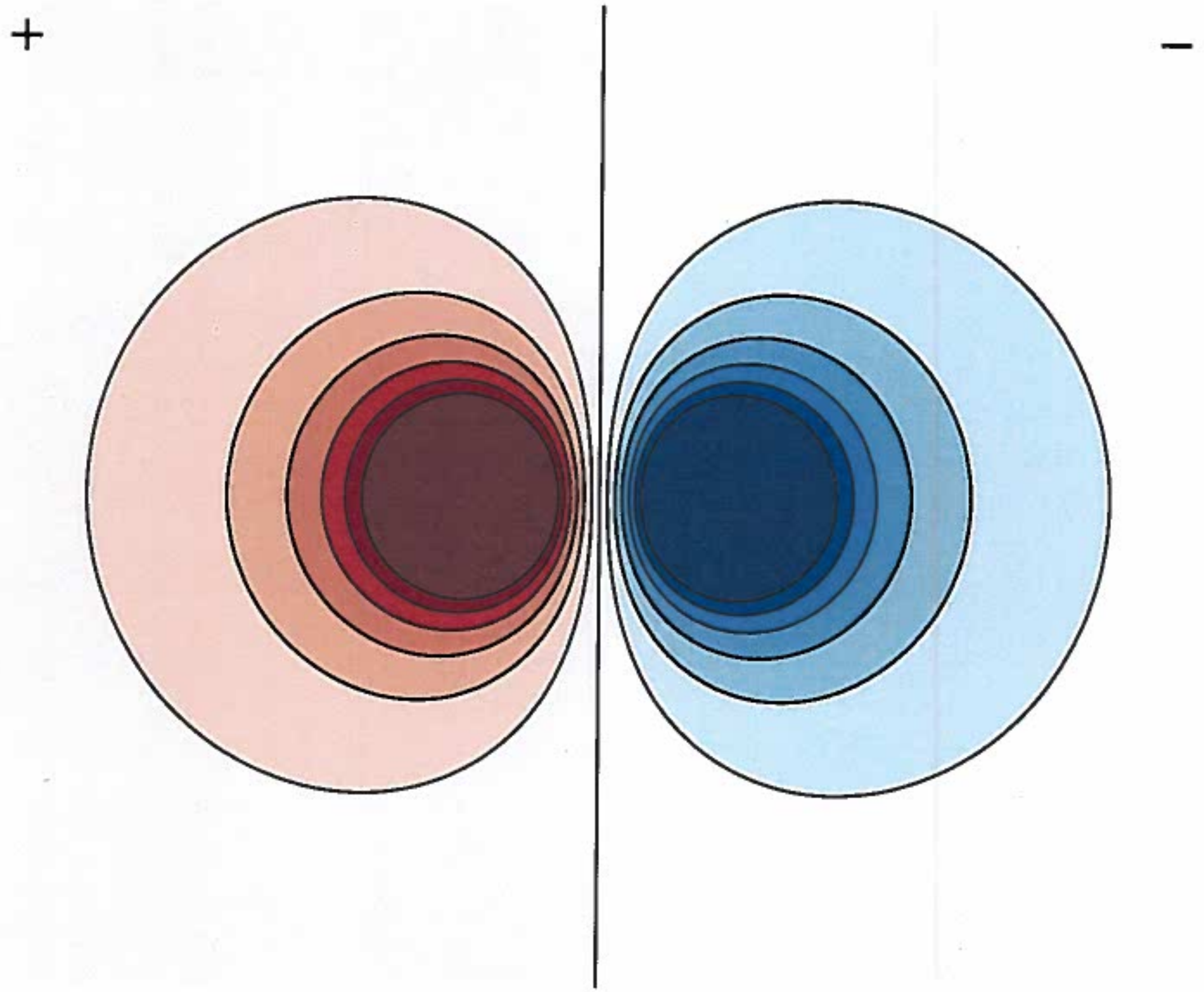
$$V(\underline{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\underline{r} - \underline{a}|} = \frac{q}{4\pi\varepsilon_0} \frac{1}{\left(r^2 + a^2 - 2ra\cos\alpha\right)^{1/2}}$$

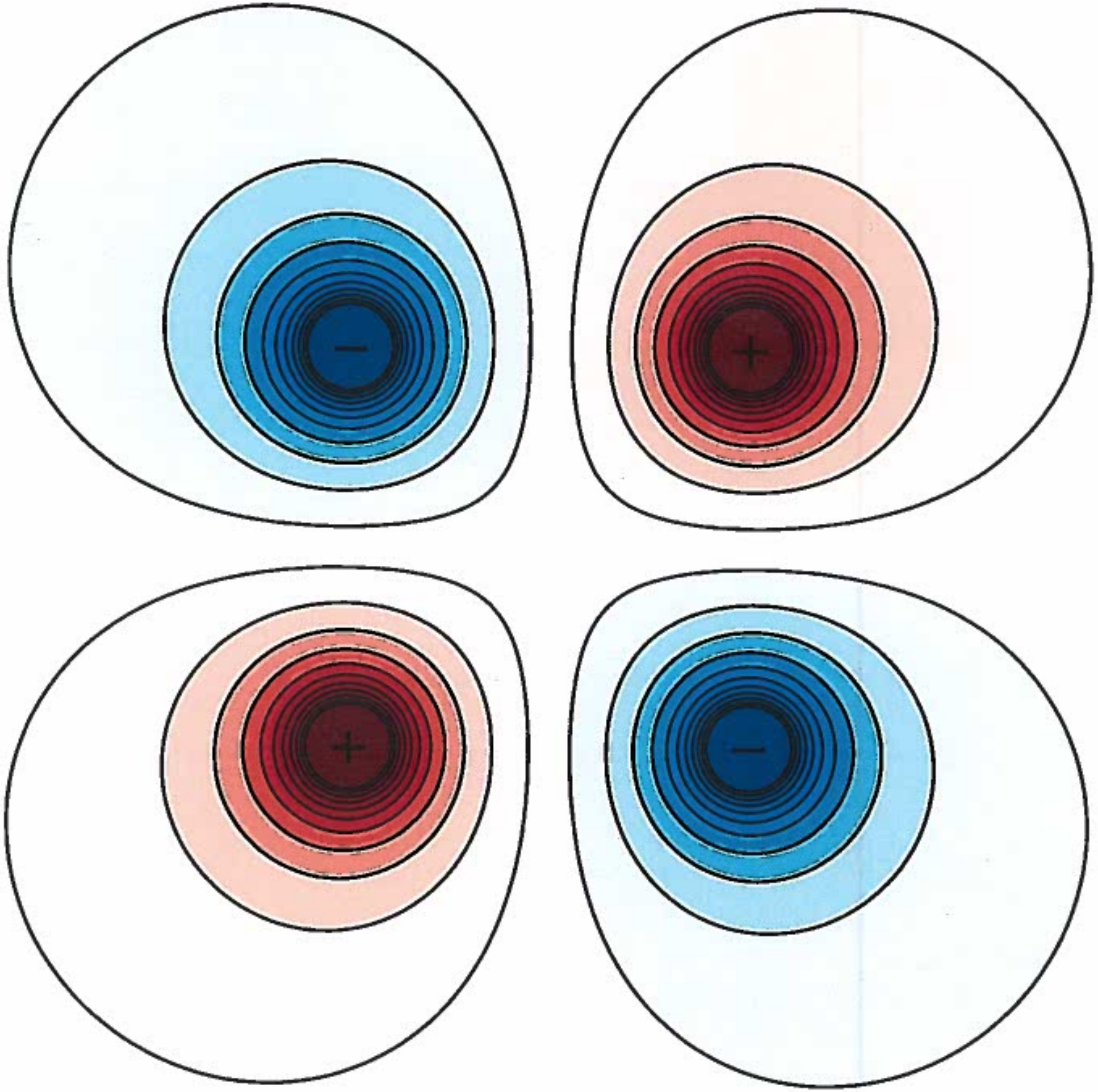
$$\approx \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \left(1 - 2\frac{a}{r}\cos\alpha\right)^{-1/2}$$

$$\approx \frac{q}{4\pi\varepsilon_0} \left(\frac{1}{r} + \frac{a\cos\alpha}{r^2} + \ldots\ldots \right)$$

"monopole" term as if charge were at the origin.

"dipole" and higher terms appear because charge is not at the origin





Suggestions for further reading: Electrostatic Multipoles

Griffiths: "Introduction to Electrodynamics" Chapter 3·4

Heald and Marion: "Classical Electromagnetic Radiation" Chapter 2

Rather more advanced level:

Jackson: "Classical Electrodynamics" Chapter 4

Suggestions for further reading on magnetostatics for the next lecture

Griffiths : "Introduction to Electrodynamics". Chapter 5.

Heald and Marion : "Classical Electromagnetic Radiation".
e.g., Sections 1.4, 1.5, 1.7, 2.6 & 2.7.

Rather more advanced level :

Jackson : "Classical Electrodynamics". Chapter 5.