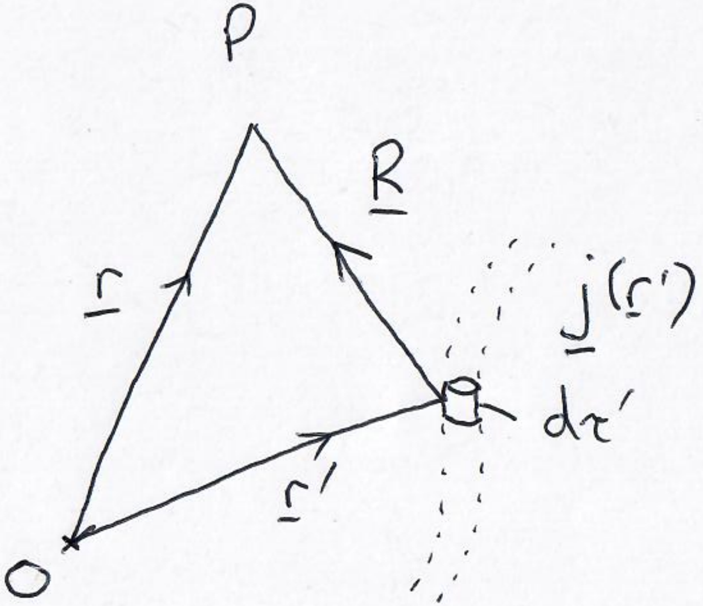
Lecture 6) Magnetostatics

What are the equivalent expressions for Magnetostatics to those we obtained in Electrostatics?

This corresponds to the regime $\frac{\partial \underline{j}}{\partial t} = 0$, $\frac{\partial \rho}{\partial t} = 0$

(current flows in closed loops)

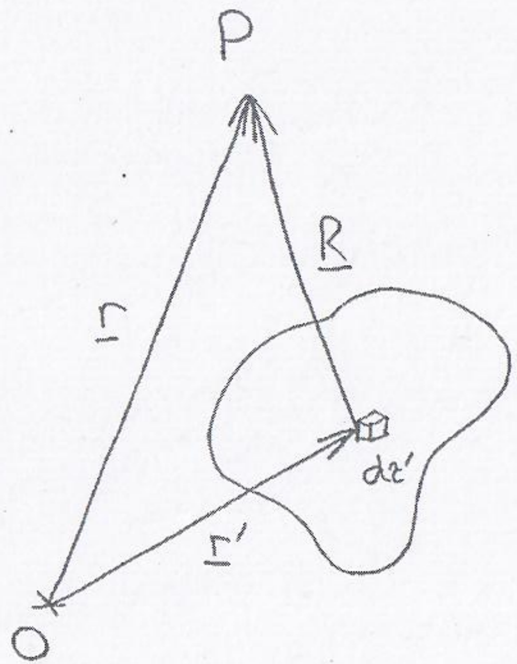


One possible starting point (in terms of the potential):

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{j}(\underline{r}')}{R}\, d\tau' \quad \text{(Eqn 6.1)}$$

Summary from Lectures 1 and 2

In **Electrostatics** we have:



$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\underline{r}')\,d\tau'}{R} \qquad \text{where } \underline{R} = \underline{r} - \underline{r}'$$

We showed that $\nabla\left(\frac{1}{R}\right) = -\frac{\hat{\underline{R}}}{R^2}$

Therefore, defining $\underline{E} = -\nabla V$ gave

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\underline{R}}}{R^2}\rho(\underline{r}')\,d\tau'$$

We showed that: $\Phi = \oint_S \underline{E}\cdot\underline{da} = \frac{1}{\varepsilon_0}\int_v \rho(\underline{r}')\,d\tau' = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$,

or in differential form $\nabla\cdot\underline{E} = \frac{\rho}{\varepsilon_0}$ or $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

and also that $\nabla\times\underline{E} = 0$

Reminders

$\underline{r}'$ position of "source" (charge, current)

$\underline{r}$ position at which potential/field evalutated (relative to origin)

$\underline{R}$ position at which potential/field evaluated (relative to "source")

$$\nabla = \left(\hat{\underline{x}} \frac{\partial}{\partial x} + \hat{\underline{y}} \frac{\partial}{\partial y} + \hat{\underline{z}} \frac{\partial}{\partial z} \right)$$

represents partial differentiation w.r.t the unprimed coordinates

$\therefore$ ∇ operating on any scalar $f(\underline{r}')$ or vector $\underline{v}(\underline{r}')$ that is a function only of the primed coordinates gives zero.

Notes

1) Contribution to $\underline{A}(\underline{r})$ from $\underline{j}(\underline{r}')\,d\tau'$ is in the same direction as $\underline{j}(\underline{r}')$

2) We can think of eqn 6.1 as effectively three equations that relate components of $A_i(\underline{r})$ to $j_i(\underline{r})$ for $i = 1, 2, 3$

3) Note the striking similarity in form of:

$$A_i(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{j_i(\underline{r}')}{R}\, d\tau' \quad \text{to} \quad V(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\underline{r}')}{R}\, d\tau' \;!$$

4) Later in the course we shall show that

$\left(\frac{V}{c}, \underline{A}\right)$ and $(\rho c, \underline{j})$ are 4-vectors and write the equations that relate them in 4-vector form.

Since $\underline{j}(\underline{r}') = \rho(\underline{r}')\,\underline{v}(\underline{r}')$ where $\underline{v}(\underline{r}')$ is the velocity

we can write also

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\rho(\underline{r}')\,\underline{v}(\underline{r}')}{R}\, d\tau'$$

makes the connection to the equation for $V(\underline{r})$ even more striking!

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$

Let's go from $\underline{A}$ to $\underline{B}$;-)

Vector Identity (7) on the "useful formula" sheet :

$$\nabla \times \left(\frac{1}{R} \underline{j}(\underline{r}') \right) = \frac{1}{R} \underbrace{\left(\nabla \times \underline{j}(\underline{r}') \right)}_{=0 \text{ because } \nabla \times f(\underline{r}') = 0} - \underline{j}(\underline{r}') \times \underbrace{\nabla\left(\frac{1}{R}\right)}_{= -\frac{\hat{\underline{R}}}{R^2} \text{ from Lecture 1}} \qquad \text{(Eqn 6.2)}$$

Combining (6.1) and (6.2) we get :

$$\nabla \times \underline{A} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\underline{j}(\underline{r}')}{R} \right) d\tau' = \frac{\mu_0}{4\pi} \int \underline{j}(\underline{r}') \times \frac{\hat{\underline{R}}}{R^2} \, d\tau' = \underline{B}$$

or, alternatively, writing $\underline{j}(\underline{r}') \, d\tau' = I \, \underline{dl}'$

$$\nabla \times \underline{A} = \frac{\mu_0 I}{4\pi} \int \underline{dl}' \times \frac{\hat{\underline{R}}}{R^2} = \underline{B}$$

Biot - Savart Law

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From vector identity (9)

$$\nabla . \underline{B} = \nabla . (\nabla \times \underline{A}) = 0$$

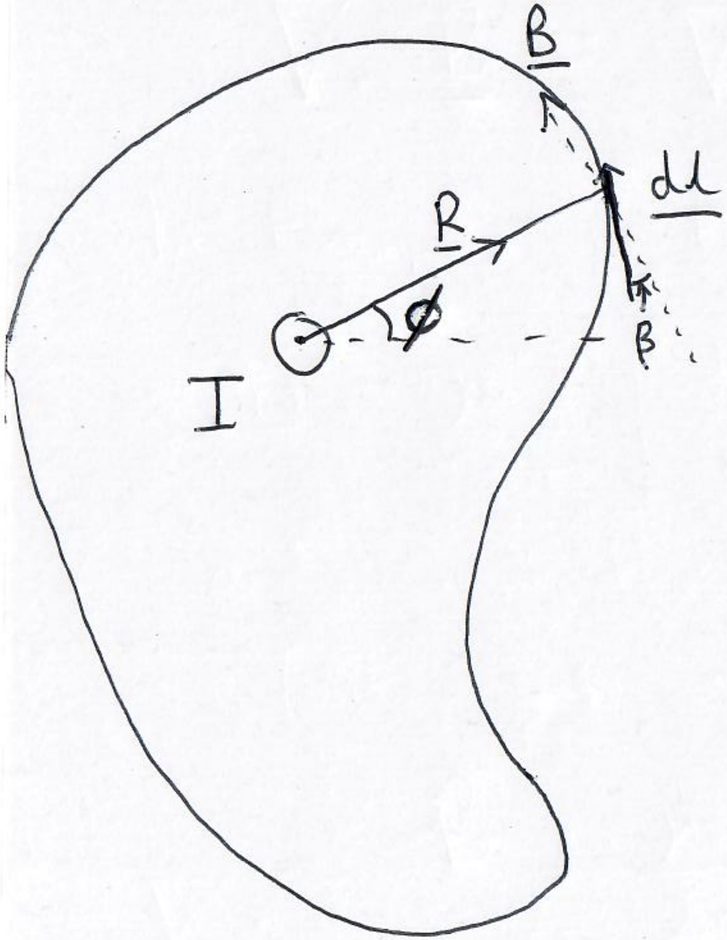
In Example Sheet 2 Q 2(b) you will start from

$$\underline{A} = \frac{\mu_0 I}{4\pi} \int \frac{dl'}{R}$$

and show that the B field at a distance R from an infinite wire carrying current I is given by:

$$B = \frac{\mu_0 I}{2\pi R}$$

Calculate circulation of $\underline{B}$ around an arbitrary closed path enclosing I



Let R and β vary arbitrarily with ϕ

Let I be $\perp$ out of the paper.

$$dl = \frac{R\,d\phi}{\cos\beta}$$

$$\underline{B}\cdot d\underline{l} = B\,dl\cos\beta$$

$$\oint \underline{B}\cdot d\underline{l} = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} \frac{1}{R}\left(\frac{R\,d\phi}{\cos\beta}\right)\cos\beta = \mu_0 I$$

N.B. We can ignore any component of $d\underline{l}$ out of the plane since $\underline{B}$ lies in the plane.

Apply Stokes' theorem

$$\int_S \nabla \times \underline{B} \, d\underline{a} = \oint \underline{B} \cdot d\underline{l} = \mu_0 I = \mu_0 \int_S \underline{j} \cdot d\underline{a}$$

can be true for all S only if

$$\nabla \times \underline{B} = \mu_0 \underline{j}$$