Lecture 7a) A more general derivation of $\nabla \times \underline{B} = \mu \underline{j}$

Point charges in electrostatics (a slight diversion)

Consider a point charge q at the origin $\underline{r}' = 0$

$$q = \int_V \rho(\underline{r}')\, d\tau'$$

can be true only if $\rho(\underline{r}') = q\,\delta^3(\underline{r}')$ (Eqn 7.1)

Poisson's equation for a point charge at position $\underline{r}'$

$$\frac{\rho(\underline{r})}{\varepsilon} = -\nabla^2 V(\underline{r}) = \frac{-q}{4\pi\varepsilon_0}\nabla^2\left(\frac{1}{R}\right) = \nabla\cdot\underline{E}(\underline{r}) = \frac{q}{4\pi\varepsilon_0}\nabla\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right) \quad \text{(Eqn 7.2)}$$

$$\therefore\ -\nabla^2\left(\frac{1}{R}\right) = \nabla\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right) = 4\pi\,\delta^3(\underline{R}) = 4\pi\,\delta^3(\underline{r}-\underline{r}') \quad \text{(Eqn 7.3)}$$

Note: $\nabla^2 V$ and $\nabla\cdot\underline{E}$ are zero everywhere except at the location of q!

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Back to Magnetostatics !

$$\nabla \times \underline{B} = \nabla \times (\nabla \times \underline{A}) = \nabla(\nabla \cdot \underline{A}) - \nabla^2 \underline{A}$$

(from vector identity (11))

Because $\underline{B} = \nabla \times \underline{A}$ we can transform

$$\underline{A} \rightarrow \underline{A} + \nabla \chi$$

where χ is any scalar function of $\underline{r}$ (from vector identity (10))

without changing $\underline{B}$.

We are free to <u>choose</u> $\nabla \cdot \underline{A} = 0$ in magnetostatics

This choice is called the "Coulomb" gauge :

$\therefore$ In this gauge $\nabla \times \underline{B} = -\nabla^2 \underline{A}$

$$= -\frac{\mu_0}{4\pi} \int \nabla^2 \left\{ \frac{\underline{j}(\underline{r}')}{R} \right\} d\tau' = -\frac{\mu_0}{4\pi} \int \underline{j}(\underline{r}') \nabla^2 \left(\frac{1}{R} \right) d\tau'$$

where $\nabla^2 \underline{j}(\underline{r}') = 0$.

$$= -\frac{\mu_0}{4\pi}\int \underline{j}(\underline{r}')\left[-4\pi\,\delta^3(\underline{r}-\underline{r}')\right]d\tau'$$

$$= \mu_0\,\underline{j}(\underline{r})$$

Since, $\underline{B}$, $\underline{A}$ and $\underline{j}(\underline{r})$ are all evaluated at the same place, $\underline{r}$, we can just write the usual expression

$$\nabla\times\underline{B} = -\nabla^2\underline{A} = \mu_0\,\underline{j}$$

Notes: 1) This is a more general derivation than that given in Lecture 6 for $\nabla\times\underline{B}$

2) We have obtained a vector version of Poisson's Equation or 3 scalar equations $-\nabla^2 A_i = \mu_0 j_i \quad i = x, y, z.$

3) Let's include explicitly the "$(\underline{r})$" into the inhomogeneous Maxwell's equations (statics)

$$\nabla \cdot \underline{E}(\underline{r}) = -\nabla^2 V(\underline{r}) = \frac{\rho(\underline{r})}{\varepsilon_0}$$

$$\nabla \times \underline{B}(\underline{r}) = -\nabla^2 \underline{A}(\underline{r}) = \mu_0 \underline{j}(\underline{r})$$

That is

$\nabla . \underline{E}$ and $\nabla^2 V$ at a given point depend only on ρ at exactly the same point

$\nabla \times \underline{B}$ and $\nabla^2 \underline{A}$ " " - - - - - - - - - - $\underline{j}$ - - - - - -

The $\rho(\underline{r}')$ for all $\underline{r}' \neq \underline{r}$ have no effect on $\nabla . \underline{E}$ and $\nabla^2 V$ at $\underline{r}$!

" $\underline{j}(\underline{r}')$ - - - - - - - - - $\nabla \times \underline{B}$ and $\nabla^2 \underline{A}$ at $\underline{r}$!

Suggestions for further reading on magnetostatics

Griffiths : "Introduction to Electrodynamics". Chapter 5.

Heald and Marion : "Classical Electromagnetic Radiation".
e.g., Sections 1.4, 1.5, 1.7, 2.6 & 2.7

Rather more advanced level:

Jackson : "Classical Electrodynamics". Chapter 5.

In particular, for the (rather long) proof that $\nabla \times \underline{B} = \mu_0 \underline{j}$ starting from Biot-Savart see, e.g.,

Griffiths. Section 5.3.2.

Jackson. Section 5.3.