A technical aside — Lecture 7(b)

Is our choice of $\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{j}(\underline{r}')}{R}\,d\tau'$ consistent with $\nabla\cdot\underline{A} = 0$?

(Coulomb gauge condition)

We need two mathematical "tricks"

1) We shall use symbol $\nabla_{r'}$ to represent differentiation with respect to the primed coordinates

$$\nabla_{r'} = \hat{\underline{x}}\frac{\partial}{\partial x'} + \hat{\underline{y}}\frac{\partial}{\partial y'} + \hat{\underline{z}}\frac{\partial}{\partial z'}$$

$\nabla_{r'} f(\underline{r}) = 0$ where $f(\underline{r})$ is a function only of the unprimed coordinates.

$$\nabla_{r'} f(\underline{r}-\underline{r}') = -\nabla f(\underline{r}-\underline{r}') \qquad \text{(Eqn 7(b).1)}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

2) Integration by parts in vector calculus

Using, e.g., vector identity (5)

$$\int_V f\,(\nabla\cdot\underline{A})\,d\tau' + \int_V \underline{A}\cdot(\nabla f)\,d\tau' = \int_V \nabla\cdot(f\underline{A})\,d\tau' = \int_S f\underline{A}\cdot d\underline{a}$$

from the divergence theorem

$$\therefore \int_V \underline{A}\cdot(\nabla f)\,d\tau = \int_S f\underline{A}\cdot d\underline{a} - \int_V f\,(\nabla\cdot\underline{A})\,d\tau' \qquad \text{(Eqn 7(b).2)}$$

A surprisingly useful technique.

- Transfered the action of ∇ from one element of a product to the other.
- at the expense of having to evaluate a "boundary integral"
  - this can be taken to be zero in cases in which the boundary is placed far away from the sources ρ, $\underline{j}$.

Returning to the question of $\nabla \cdot \underline{A}$

$$\frac{4\pi}{\mu_0}\left(\nabla \cdot \underline{A}\right) = \int_v \underline{j}(\underline{r}') \cdot \nabla\left(\frac{1}{R}\right) d\tau' = -\int_v \underline{j}(\underline{r}') \cdot \nabla_{r'}\left(\frac{1}{R}\right) d\tau'$$

from eqn (6.1) and $\nabla \cdot j(r') = 0$

from eqn 7.(b).1

$$= -\underbrace{\int_S \left[\underline{j}(\underline{r}')\left(\frac{1}{R}\right)\right] \cdot \underline{da}}_{=0} + \underbrace{\int_v \left[\nabla_{r'} \cdot \underline{j}(\underline{r}')\right] \frac{1}{R}\, d\tau'}_{=0}$$

from eqn 7.(b).2

$= 0$ by taking the boundary to be infinitely far from any $\underline{j}(\underline{r}')$ in our problem

$= 0$ because $\left.\begin{aligned}\frac{\partial\rho}{\partial t} &= 0\\ \nabla \cdot \underline{j} &= 0\end{aligned}\right\}$ are the conditions for magnetostatics

$$\therefore \quad \nabla \cdot \underline{A} = 0 \quad \text{as required}$$