Lecture 7(c): Multipole expansions in Magnetostatics

Given similarity in form between $V(\underline{r})$ in terms of $\rho(\underline{r}')$

$\underline{A}(\underline{r})$ " " " $\underline{j}(\underline{r}')$

we can just write down the main results!

$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\underline{l}'}{R} \quad \text{and} \quad \frac{1}{R} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^{l} P_l(\cos\alpha)$$

Consider current flowing in a closed loop

$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi}\Bigg[\underbrace{\frac{1}{r}\oint d\underline{l}'}_{l=0\ \text{"monopole" term}\ =0} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\, d\underline{l}'}_{\text{"dipole" term}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left[\frac{3\cos^2\alpha - 1}{2}\right] d\underline{l}'}_{\text{"quadrupole" term}}$$

$\because$ Integral of vector displacement around a closed path $= 0$

Notes

1) Contribution to $\underline{A}(\underline{r})$ from $\underline{j}(\underline{r}')\,d\tau'$ is in the same directio
as $\underline{j}(\underline{r}')$

2) We can think of eqn 6.1 as effectively three equations that relate components of $A_i(\underline{r})$ to $j_i(\underline{r}')$ for $i=1,2,$

3) Note the striking similarity in form of:

$$A_i(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{j_i(\underline{r}')}{R}\,d\tau' \quad \text{to} \quad V(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\underline{r}')}{R}\,d\tau' \,!$$

4) Later in the course we shall show that

$\left(\frac{V}{c}, \underline{A}\right)$ and $(\rho c, \underline{j})$ are 4-vectors and write the equation that relate them in 4-vector form.

Let's take a closer look at the dipole term

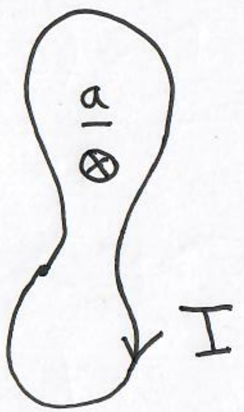
In example sheet 3 you will show that

$$\oint r' \cos\alpha \, \underline{dl}' = \oint (\hat{\underline{r}} \cdot \underline{r}') \, \underline{dl}' = -\hat{\underline{r}} \times \int_S \underline{da}' = -\hat{\underline{r}} \times \underline{a}'$$

"vector area"

$$\therefore \quad \underline{A}_{\text{dipole}}(\underline{r}) = \frac{\mu_0 I}{4\pi} \frac{\underline{a} \times \hat{\underline{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{\underline{r}}}{r^2}$$

where $\underline{m} = I \underline{a}$ is the "magnetic dipole moment"

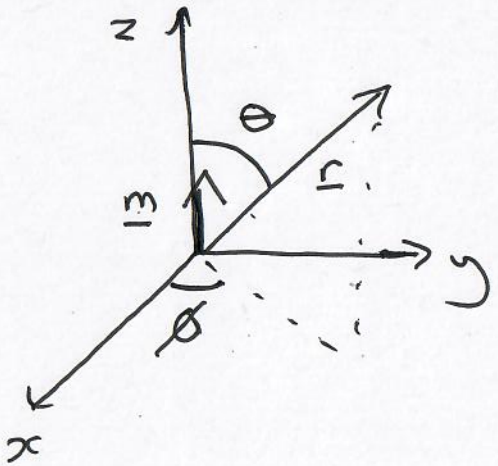


__Note:__ If the loop lies in a plane $|\underline{a}|$ is the area enclosed

Direction of $\underline{a}$ is given by the circulation of I using the right hand rule.

For example, lets consider $\underline{m} = m\hat{\underline{z}}$ at the origin.

and work out expressions for $\underline{A}$ and $\underline{B}$



$$\underline{A}_{dipole} = \frac{\mu_0 m}{4\pi} \frac{\sin\theta}{r^2} \hat{\underline{\phi}}$$

$$\underline{B} = \nabla \times \underline{A} = \frac{\mu_0 m}{4\pi}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\sin\theta}{r^2}\right)\hat{\underline{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\sin\theta}{r^2}\right)\hat{\underline{\theta}}\right]$$

$$= \frac{\mu_0 m}{4\pi}\left[\frac{2\cos\theta}{r^3}\hat{\underline{r}} + \frac{\sin\theta}{r^3}\hat{\underline{\theta}}\right]$$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$