Lecture 9a) The Wave Equations for the Potentials

Start from the inhomogeneous Maxwell Equations.

Substitute for the fields in terms of the potentials

$$\underline{E} = -\nabla V - \frac{\partial \underline{A}}{\partial t} \quad \text{and} \quad \underline{B} = \nabla \times \underline{A}$$

A) $$\nabla . \underline{E} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla . \underline{A}) = \frac{\rho}{\varepsilon_0} \qquad \text{(Eqn 9.1)}$$

B) $$\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$$\text{L.H.S.} = \nabla \times (\nabla \times \underline{A}) = \nabla(\nabla . \underline{A}) - \nabla^2 \underline{A} \qquad \text{[vector identity (11)]}$$

$$\text{R.H.S.} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \left[-\nabla\left(\frac{\partial V}{\partial t}\right) - \frac{\partial^2 \underline{A}}{\partial t^2} \right]$$

$$\therefore \quad \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} - \nabla^2 \underline{A} + \nabla\left(\mu_0 \varepsilon_0 \frac{\partial V}{\partial t} + \nabla . \underline{A} \right) = \mu_0 \underline{j} \qquad \text{(Eqn 9.2)}$$

In Electrodynamics it is convenient to choose the "Lorenz gauge"

$$\boxed{\mu_0 \varepsilon_0 \frac{\partial V}{\partial t} + \nabla \cdot \underline{A} = 0} \qquad (\text{Eqn } 9.3)$$

Inserting (9.3) into (9.1) and (9.2) we obtain

$$\mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho}{\varepsilon_0} \qquad (\text{Eqn } 9.4)$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} - \nabla^2 \underline{A} = \mu_0 \underline{j} \qquad (\text{Eqn } 9.5)$$

The <u>inhomogeneous wave equations</u> for V and $\underline{A}$

↑ because they include the sources ρ, $\underline{j}$

Since $\varepsilon_0 \mu_0 = \frac{1}{c^2}$ the potentials (and fields) propagate at the speed of light

Defining $\Box^2 \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ The "D'Alembertian"

$$\Box^2 V = \frac{\rho}{\varepsilon_0} \qquad \text{(Eqns 9.6)}$$

$$\Box^2 \underline{A} = \mu_0 \underline{j} \qquad \text{(Eqn 9.7)}$$

Notes on the wave equations for the potentials

- As in magnetostatics / electrostatics
  - Pleasing symmetry between the wave equations for V and $\underline{A}$
  - ~ "4-dimensional version" of Poisson's equations.
  - Equation for $\underline{A}$ can be thought of as three equations relating the individual components of $\underline{A}$ and $\underline{j}$

$$\Box^2 A_i = \mu_0 j_i \quad , \text{ where } i = 1,2,3 \text{ or } x,y,z .$$

- These equations contain essentially the same information as Maxwell's equations for $\underline{E}$ and $\underline{B}$
  - but in a convenient 4-component form $\left\{ \begin{array}{l} \text{as opposed} \\ \text{to 6-component} \\ \text{form for } \underline{E} \text{ and } \underline{B} \end{array} \right)$

Crying out to be written in 4-vector form!!

- If ρ and $\underline{j}$ are zero we obtain the "homogeneous" wave equations

$$\Box^2 V = 0$$

$$\Box^2 \underline{A} = 0$$

- If there is no time-dependence we recover Poisson's Equations for electro- and magneto- statics for V and $\underline{A}$.