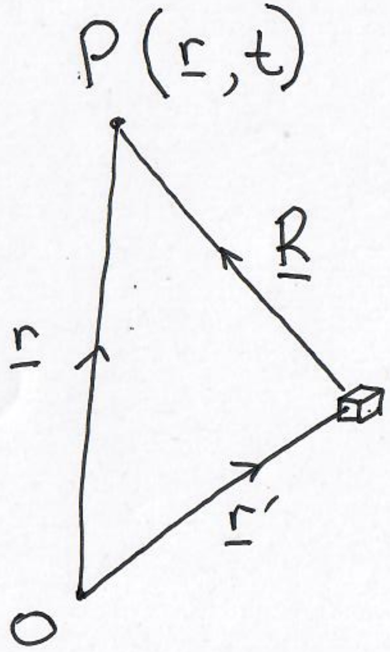
Lecture 9b) Solutions to the Wave Equations for the Potentials

E.g., in electrostatics : $V(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho(\underline{r}')}{R}\,d\tau'$ is a solution to Poisson's Equation $\nabla^2 V = -\rho/\varepsilon_0$

Take into account the time-varying ρ, $\underline{j}$



$(\rho(\underline{r}', t_{ret}))$ $\quad t_{ret} = t - \frac{R}{c}$

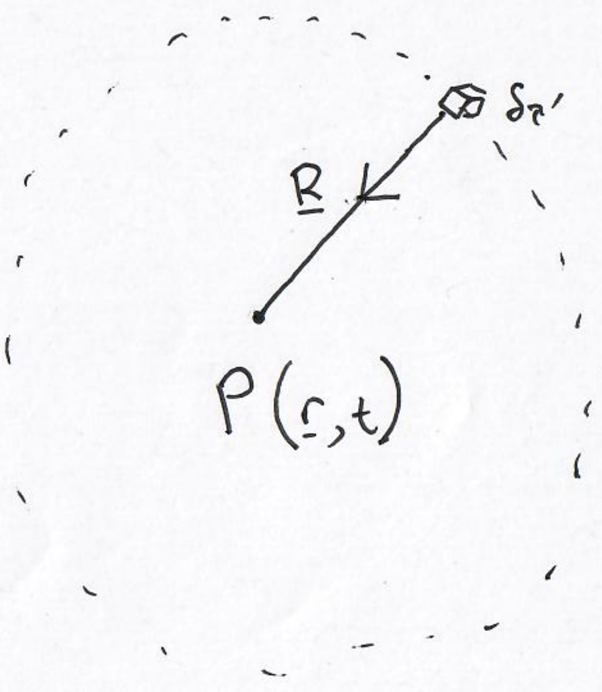
t_{ret} is called the retarded time

Guess

$$V(\underline{r}, t) = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho(\underline{r}', t_{ret})}{R}\,d\tau' \qquad \text{(Eqn 9.8)}$$

Also $\underline{A}(\underline{r},t) = \frac{\mu_0}{4\pi}\int_v \frac{\underline{j}(\underline{r}',t_{ret})}{R}\,d\tau'$ (Eqn 9.9)

Proof that the proposed $V(\underline{r}, t)$ is indeed a solution to the wave equation $\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \rho/\varepsilon_0$



In electrostatics $\nabla^2 V = -\rho/\varepsilon_0$ tells us that a particular point in space $P(\underline{r})$, $\nabla^2 V(r)$ is determined entirely by $\rho'(\underline{r})$ at exactly the <u>same point in space</u>. In electrostatics the value of $\rho(\underline{r}')$ at all other points $\underline{r}' \neq \underline{r}$ has no effect on $\nabla^2 V(\underline{r})$!

Consider the integral for $\nabla^2 V$ as the sum over an infinitessimally small volume ν_1 containing P plus a series of spherical shells centred at P that together make a second volume ν_2.

$$V = V_1 + V_2$$

Lecture 7a) A more general derivation of $\nabla \times \underline{B} = \mu \underline{j}$

Point charges in electrostatics (a slight diversion)

Consider a point charge q at the origin $\underline{r}' = 0$

$$q = \int_V \rho(\underline{r}')\, d\tau'$$

can be true only if $\rho(\underline{r}') = q\, \delta^3(\underline{r}')$ (Eqn 7.1)

Poisson's equation for a point charge at position $\underline{r}'$

$$\frac{\rho(\underline{r})}{\varepsilon} = \boxed{-\nabla^2 V(\underline{r}) = \frac{-q}{4\pi\varepsilon_0} \nabla^2\left(\frac{1}{R}\right)} = \nabla \cdot \underline{E}(\underline{r}) = \frac{q}{4\pi\varepsilon_0} \nabla \cdot \left(\frac{\hat{\underline{R}}}{R^2}\right) \quad \text{(Eqn 7.2)}$$

$$\therefore \quad \boxed{-\nabla^2\left(\frac{1}{R}\right) = \nabla \cdot \left(\frac{\hat{\underline{R}}}{R^2}\right) = 4\pi\, \delta^3(\underline{R}) = 4\pi\, \delta^3(\underline{r} - \underline{r}')} \quad \text{(Eqn 7.3)}$$

Note: $\nabla^2 V$ and $\nabla \cdot \underline{E}$ are zero everywhere except at the location of q!

Since $\mathcal{V}_1$ is infinitessimally small, we can ignore the effects of retarded time and write

$$V_1(\underline{r}, t) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}_1} \frac{\rho(\underline{r}', t)}{R}\, d\tau'$$

This is a solution to Poisson's equation $\nabla^2 V_1 = -\rho/\varepsilon_0$ (Eqn 9.10)

Now let's consider $\nabla^2 V_2$ in the region $\mathcal{V}_2$

$$\nabla^2 V_2 = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}_2} \nabla^2\left[\frac{\rho(\underline{r}', t_{ret})}{R}\right] d\tau' \qquad \text{(Eqn 9.11)}$$

N.B. The dependence of $\left[\frac{\rho(\underline{r}', t_{ret})}{R}\right]$ on the unprimed coordinates comes through the dependence on R of $t_{ret} = t - R/c$ as well as $\frac{1}{R}$

For each infinitessimal element $\delta\tau'$ we can consider spherical polar coordinates centred at $\underline{r}'$.

Since the contribution to V depends only on $R = |\underline{R}|$ and not on directions we can ignore θ, ϕ dependence in writing down ∇^2 and consider only $\frac{\partial}{\partial R}$ terms.

$$\nabla^2 = \nabla_R^2 \qquad \text{where} \quad \underline{R} = \underline{r} - \underline{r}' \quad \text{and } \underline{r}' \text{ is a constant.}$$

$$\nabla^2\left[\frac{\rho}{R}\right] = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\left[\frac{\rho}{R}\right]\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\left\{\frac{1}{R}\frac{\partial\rho}{\partial R} + \rho\left(\frac{-1}{R^2}\right)\right\}\right)$$

$$= \frac{1}{R^2}\left(\frac{\partial\rho}{\partial R} + R\frac{\partial^2\rho}{\partial R^2} - \frac{\partial\rho}{\partial R}\right) = \frac{1}{R}\frac{\partial^2\rho}{\partial R^2} \qquad \text{(Eqn 9.12)}$$

Since $\rho = \rho(t_{ret}) = \rho\left(t - \frac{R}{c}\right)$ it will be a solution to the 1-D wave equation

$$\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 \rho}{\partial R^2} \qquad (\text{Eqn } 9.13)$$

Exercise for you: Verify explicitly that this is the case!

Substituting for (9.13) into (9.12) and then into (9.11) we obtain

$$\nabla^2 V_2 = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}_2} \frac{1}{R}\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2}\, d\tau' = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underbrace{\left[\frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}_2} \frac{\rho(\mathbf{r}', t_{ret})}{R}\, d\tau'\right]}_{= V(\mathbf{r},t) \text{ as } \mathcal{V}_1 \text{ becomes infinitesimally small}}$$

[Since R is not a function of t]

$= V(\mathbf{r}, t)$ as $\mathcal{V}_1$ becomes infinitesimally small

$\mathcal{V}_2 \rightarrow V$, the entire space

$\therefore$ we can write

$$\nabla^2 V_2 = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \qquad (\text{Eqn } 9.14)$$

Adding equations (9.10) and (9.14)

$$\nabla^2 (V_1 + V_2) = \nabla^2 V = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \frac{\rho}{\varepsilon_0}$$

which demonstrates that our guess for the form of V (eqn 9.8) is, indeed, a solution to the wave equation.

A similar proof works for each component of A_i in terms of the relevant j_i.