PHYS30441 Electrodynamics: Revision Example Sheet

1. (a) Consider the pre-Maxwell equations: Ampere's Law ($\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$) and Gauss' Law. Show that these are *inconsistent* with charge conservation, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

(b) Show that if instead the full Maxwell equation for $\nabla \times \mathbf{B}$, including the displacement current, is used together with Gauss' Law the inconsistency is removed. In other words, we can derive the charge conservation equation from Maxwell's equations.

2. (a) Demonstrate that the following vector potential is consistent with a homogenous magnetic field:

$$\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}.$$

Hint: you may find the following identity useful:
$\nabla \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\nabla \cdot \mathbf{w}) - \mathbf{w}(\nabla \cdot \mathbf{v}) + (\mathbf{w} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{w}$.

N.B. "homogeneous" means that $\mathbf{B}$ has the same magnitude and direction at all points in space.

(b) Verify that the vector potential given above satisfies the Coulomb gauge. That is, $\nabla \cdot \mathbf{A} = 0$.

3. (a) An electrical discharge has an electric field of 10^7 Vm^{-1}. Calculate the energy density.

(b) At the centre of the ATLAS experiment at the CERN LHC is a superconducting solenoidal magnet. This magnet is 5.3 m long and has a radius of 1.25 m; it has around 1100 turns, each carrying a current of 7700 A. Using the approximation that this can be considered as an "ideal" or "infinitely long" solenoid, show that the magnetic field produced by the solenoid is around 2.0 T. Calculate the stored magnetic energy, ignoring any fields outside the cylindrical volume of the solenoid.

4. A flat disk of radius R, carrying a uniform surface charge density σ, is rotating at constant angular velocity ω about an axis that is normal to the surface and passes through its centre. Find its magnetic dipole moment.

5. (a) Using Maxwell's equations show that the electric field $\mathbf{E}$ can be written in terms of a vector and scalar potential:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.$$

(b) Also show that in a region free of charge and current, the electric and magnetic fields in vacuum satisfy the wave equations:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \qquad \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0.$$

6. Suppose we have a vector potential $\mathbf{A}$ and scalar potential V. Show that replacing these with new potentials $\mathbf{A} \to \mathbf{A} + \nabla g$, $V \to V - \partial g/\partial t$ (where g is any scalar function) gives

exactly the same electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. This is called a 'gauge transformation'.

7. Consider the vector potential

$$\mathbf{A} = B_0\{\frac{yz}{a}\sin\omega t\hat{\mathbf{y}} + (\frac{x^3}{a^2} + 2z)\cos\omega t\hat{\mathbf{z}}\},$$

where B_0, a and ω are constants.

(a) Find an expression for a scalar potential V which satisfies the Lorenz gauge condition. Why is this expression for V not unique? (You can choose any suitable solution).

(b) Calculate the corresponding electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

8. Prove the following results in vector calculus:

(a) The curl of a gradient is always zero, that is,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} V) = 0,$$

where V is a scalar field.

(b) The divergence of a curl is always zero, that is,

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0,$$

where $\boldsymbol{A}$ is a vector field.

(c)

$$\boldsymbol{\nabla}(fg) = f(\boldsymbol{\nabla} g) + g(\boldsymbol{\nabla} f),$$

where f and g are a scalar fields.

(d)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{f}\boldsymbol{A}) = f(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + (\boldsymbol{\nabla} f) \cdot \boldsymbol{A},$$

where f is a scalar field and $\boldsymbol{A}$ is a vector field.

Notes:

- *These proofs are easiest to establish in cartesian coordinates. (Once proven in cartesian coordinates they must be valid for all coordinate systems.)*
- *You may wish to write down these proofs in **two** ways: (i) picking one of the cartesian coordinates, say x, and then writing out explicitly all terms contributing to the x coordinate of the vector expression to be evaluated, and (ii) using general index notation for 3-dimensional vectors.*

9. A rod of length L' lies at rest along the x' axis of a frame of reference S'. Relative to a second inertial frame S, the frame S' moves in the x direction with constant speed β (in units of c). At the time $t = 0$ the rear end of the moving rod passes an observer at rest in S at the origin $x = 0$. *(Hint: you may find it convenient to define this 'event' to occur at $t' = 0, x' = 0$ in frame S'.)* Simultaneously in S the front end of the rod passes a second observer at rest in S at the coordinate x_1.

(a) Using an appropriate Lorentz transformation equation write an expression for x_1 in terms of L'.

(b) Comment on the physical significance of the expression you have obtained above for x_1.

(c) According to an observer in S', at what time t'_1 is the measurement of the front end of the rod made?

(d) Using an appropriate Lorentz transformation equation find the distance, as measured in S', between the points $x = 0$ and $x = x_1$.

10. A charged pion π^- of mass m_π is at rest. It decays to produce a muon μ^- of (rest) mass m_μ and a (massless) neutrino.

    Find, in terms of m_π and m_μ, (a) the energy and (b) the speed of the μ^-.