Electrodynamics (PHYS30441) Special Relativity Revision

Preamble:

In this course we shall (starting in Lecture 11) be describing 4-vectors in Special Relativity using index notation. However, you do NOT need to be familiar with index notation for 4-vectors in order to do the following revision problems!

For these problems all you need to know, beyond your first year Quantum Physics and Relativity course (PHYS10121) is that we can use the following shorthand notation:

$$\underset{\sim}{x} = x^\mu = \left[x^0, x^1, x^2, x^3\right] = [ct, x, y, z] = [ct, \boldsymbol{r}]$$

for the "space-time" 4-vector[1].

In this shorthand we can write the Lorentz transformation

$$\begin{aligned} x'^0 &= \gamma\left(x^0 - \beta x^1\right) \\ x'^1 &= \gamma\left(-\beta x^0 + x^1\right) \\ x'^2 &= x^2 \\ x'^3 &= x^3, \end{aligned}$$

where the inertial frame of reference S' (primed coordinates) moves relative to frame S (unprimed coordinates) in the x^1 direction with constant speed[2] $\beta = v/c$ and

$$\gamma = \frac{1}{\left(1 - (v/c)^2\right)^{\frac{1}{2}}} = \frac{1}{\left(1 - \beta^2\right)^{\frac{1}{2}}}.$$

If this all seems too unfamiliar to you, then just translate the following problems using $x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$ and solve the problems in terms of $[t, x, y, z]$.

N.B. In preparation for using index notation for 4-vectors, I would very strongly encourage you to practice using index notation for 3-dimensional vectors, e.g., by working through the vector calculus revision problem sheet using vector notation.

Problems:

1. (a) Look up in your notes for Quantum Physics and Relativity (PHYS10121) the expressions for the Lorentz transformation in terms of $[t, x, y, z]$. By applying the translation $x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$ verify explicitly the expressions given in the preamble above for the Lorentz transformation in terms of $[x^0, x^1, x^2, x^3]$.

   (b) Verify explicitly using the Lorentz transformations that the space-time interval between the coordinates of two events (1) and (2): $\Delta x^\mu = x^\mu_{(1)} - x^\mu_{(2)} = (c\Delta t, \Delta\boldsymbol{r})$, is also a 4-vector.

2. Verify explicitly using the Lorentz transformations that the scalar product between two 4-vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, which is equal to $a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3$, is a Lorentz invariant.

3. Two events in the intertial frame of reference S are connected by a signal travelling at the speed of light; that is $\Delta r = c\Delta t$. verify explicitly using the Lorentz transformations[3] that this will be the case also for the inertial frame S', which is moving with constant speed $\beta = v/c$ relative to frame S; that is $\Delta r' = c\Delta t'$.

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[1] Note that we define the "zeroth" component of the 4-vector to be ct, rather than just t, so that all four components of the 4-vector have the same units [length]. Intuitively, you might find it useful to think of x^0 as "the distance that a signal travelling at the speed of light would travel in time t".

[2] Alternatively, we can think of β as "the speed in units of c".

[3] This is a more long-winded alternative to the proof given in Lecture 10.

4. Starting from the expressions given in the preamble above for the Lorentz transformation in terms of $[x^0, x^1, x^2, x^3]$, demonstrate explicitly that the inverse Lorentz transformation for the unprimed coordinates in terms of the primed coordinates is given by

$$x^0 = \gamma\,(x'^0 + \beta x'^1)$$
$$x^1 = \gamma\,(\beta x'^0 + x'^1)$$
$$x^2 = x'^2$$
$$x^3 = x'^3.$$

5. The aim of this problem is to challenge your intuitive understanding of "time dilation" and to reinforce the idea of "proper time". Proper time will be crucial when we come to define some of the 4-vectors we shall need in Electrodynamics.

   A muon is produced in the upper atmosphere and travel towards the surface of the earth at a constant high speed. Let the muon be at rest in a frame S' and let an observer at the surface of the earth be at rest in a frame S. Let frame S' be moving relative to frame S in the x^1 direction with constant speed β. (That is, for the purposes of this problem the x^1 direction points vertically *downwards*.) Let the space-time coordinates of the point at which the muon is produced be given by $x'^0 = x^0 = 0$ and $x'^1 = x^1 = 0$.

   (a) From the point of view of the observer at the surface of the earth find the time that elapses between the production of the muon and its arrival at the surface of the earth (i) in frame S and (ii) in frame S'. Thus explain how, even though the muon lifetime is around 3.2×10^{-6} s, there is a high probability that it will arrive at the surface of the earth before decaying.

   (b) So far, all perfectly straightforward! However, now let's consider the elapsed times from the point of view of the muon. The principle of relativity tells us that from the point of view of the muon the clock at the surface of the earth in frame S is moving and therefore runs more slowly than the clock at rest with respect to the muon. Therefore, from the point of view of the muon the time that elapses in its own rest frame S' between its production and arrival is **greater** than that measured by the clock at the surface of the earth. Therefore, in its own rest frame shouldn't the muon's lifetime should be **greater** than that measured by the clock at the surface of the earth? Thus, from the point of view of the clock at the surface of the earth, shouldn't the muon decay more quickly than naively be expected and therefore never arrive at the surface of the earth?

   Look at your 1st year Special Relativity notes (PHYS10121) or a text book and write down in your own words an explanation of the concept of "proper time".

   (c) Use the concept of proper time to explain why there is an asymmetry in the views of the problem, as given in (a) and (b) above, and to explain which is correct.

   Additional hint: when dealing with apparent paradoxes or inconsistencies in special relativity, it can be helpful to analyse a problem in terms of distinct space time events that have definite space-time coordinates. Then use the Lorentz transformations to relate the space-time coordinates in one frame of reference to those in other frames.

6. Consider a particle that is moving in frame S' with speed (in units of c) β_1 in the x^1 direction. Frame S' is moving with respect to frame S in the x^1 direction with speed β_2. Find an expression in terms of β_1 and β_2 for β_3, the speed (in units of c) of the particle in frame S.