Q1 (a)

Lorentz transformation equations from PHYS10121

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad (1)$$

$$x' = \gamma(x - vt) \qquad (2)$$

$$y' = y \qquad (3)$$

$$z' = z. \qquad (4)$$

Using $x^0 = ct$, $x^1 = x$ we can write equation (1) as:

$$\frac{x'^0}{c} = \gamma\left(\frac{x^0}{c} - \beta\frac{x^1}{c}\right)$$

or $x'^0 = \gamma(x^0 - \beta x^1)$; as required.

Similarly for equation (2):

$$x'^1 = \gamma\left(x^1 - v\frac{x^0}{c}\right)$$

or $x'^1 = \gamma(-\beta x^0 + x^1)$, as required.

The other two equations follow trivially from the definitions $x^2 = y$, $x^3 = z$.

Q1) (b) $x'^{0}_{(i)} = \gamma\left(x^{0}_{(i)} - \beta x^{1}_{(i)}\right)$

$$x'^{1}_{(i)} = \gamma\left(-\beta x^{0}_{(i)} + x^{1}_{(i)}\right)$$

$$x'^{2}_{(i)} = x^{2}_{(i)}$$

$$x'^{3}_{(i)} = x^{3}_{(i)}$$

$$\therefore \quad \Delta x'^{0} = x'^{0}_{(1)} - x'^{0}_{(2)} = \gamma\left(\left[x^{0}_{(1)} - x^{0}_{(2)}\right] - \beta\left[x^{1}_{(1)} - x^{1}_{(2)}\right]\right)$$

$$= \gamma\left(\Delta x^{0} - \beta \Delta x^{1}\right)$$

$$\Delta x'^{1} = x'^{1}_{(1)} - x'^{1}_{(2)} = \gamma\left(-\beta\left[x^{0}_{(1)} - x^{0}_{(2)}\right] + \left[x^{1}_{(1)} - x^{1}_{(2)}\right]\right)$$

$$= \gamma\left(-\beta \Delta x^{0} + \Delta x^{1}\right)$$

$$\Delta x'^{2} = x'^{2}_{(1)} - x'^{2}_{(2)} = x^{2}_{(1)} - x^{2}_{(2)} = \Delta x^{2}$$

and similarly for $\Delta x'^{3} = \Delta x^{3}$

$\therefore$ Δx^{μ} obeys the Lorentz Transformations and is a 4-vector.

Q2) $\underset{\sim}{a}' \cdot \underset{\sim}{b}' = a'^0 b'^0 - a'^1 b'^1 - a'^2 b'^2 - a'^3 b'^3$

$$= \gamma^2 (a^0 - \beta a^1)(b^0 - \beta b^1) - \gamma^2 (-\beta a^0 + a^1)(-\beta b^0 + b^1) - a^2 b^2 - a^3 b^3$$

$$= \gamma^2 (a^0 b^0 - a^1 b^1)(1 - \beta^2) - a^2 b^2 - a^3 b^3$$

$$= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

$= \underset{\sim}{a} \cdot \underset{\sim}{b}$, which is, therefore, an invariant.

Q3) Signal travelling at light speed implies

$$\frac{(\Delta r)^2}{(c\Delta t)^2} = \frac{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}{(\Delta x^0)^2} = 1 \quad \text{(Eqn. A)}$$

In frame S'

$$(\Delta r')^2 = \gamma^2\left(\beta^2(\Delta x^0)^2 + (\Delta x^1)^2 - 2\beta\Delta x^0\Delta x^1\right) + (\Delta x^2)^2 + (\Delta x^3)^2$$

$$= \gamma^2\left(\beta^2\left[(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)\right] + (\Delta x^1)^2 - 2\beta\Delta x^0\Delta x^1\right) + (\Delta x^2)^2 + (\Delta x^3)^2$$

{using (A)}

$$= \gamma^2(1+\beta^2)(\Delta x^1)^2 + (1+\gamma^2\beta^2)\left[(\Delta x^2)^2 + (\Delta x^3)^2\right] - 2\beta\Delta x^0\Delta x^1\gamma^2$$

$$= \gamma^2\left[(1+\beta^2)(\Delta x^1)^2 + (\Delta x^2) + (\Delta x^3)^2 - 2\beta\Delta x^0\Delta x^1\right] \quad \text{(Eqn. B)}$$

$$\left\{\text{since } (1+\gamma^2\beta^2) = 1 + \frac{\beta^2}{1-\beta^2} = \gamma^2\right\}$$

$$(\Delta x'^0)^2 = \gamma^2\left((\Delta x^0)^2 + \beta^2(\Delta x^1)^2 - 2\beta\Delta x^0\Delta x^1\right)$$

$$= \gamma^2\left[(1+\beta^2)(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 - 2\beta\Delta x^0\Delta x^1\right] \quad \text{(Eqn. C)}$$

{using (A)}

Comparing (B) and (C) shows that $(\Delta r')^2/(\Delta x'^0)^2 = 1$

$\therefore$ Signal travels at speed c in S'.

NR Proof using invariant interval is much shorter!

Q4)

$$\gamma\left(x'^0 + \beta x'^1\right)$$

$$= \gamma\left(\gamma\left[x^0 - \beta x^1\right] + \beta\gamma\left[-\beta x^0 + x^1\right]\right)$$

{using the Lorentz transformations}

$$= \gamma^2\left(1-\beta^2\right)x^0 = x^0 \quad \text{as required}$$

$$\gamma\left(\beta x'^0 + x'^1\right)$$

$$= \gamma\left(\beta\gamma\left[x^0 - \beta x^1\right] + \gamma\left[-\beta x^0 + x^1\right]\right)$$

$$= \gamma^2\left(1-\beta^2\right)x^1 = x^1 \quad \text{as required}$$

Q5) Concepts such as time dilation and length contraction can be very useful to help build an intuitive understanding of simple problems in Special Relativity. However, unless we are very careful, our intuition can occasionally lead us astray, especially when a malicious professor tries to lead us up the garden path, as in part (b) of this question.

When I get confused I like to remind myself that "the Lorentz Transformations never lie!". If we break a problem down into a set of well-defined space-time events then the L.T. allow us to work out the correct space-time coordinates in all relevant frames of reference.

Let's take this approach here!

a) Picture of the events from frame S point of view

Event 1: Production of the muon

$$t_1 = t_1' = 0 \quad , \quad x_1 = x_1' = 0$$

(NB. In this problem I use subscripts to label the events. These are not covariant indices ;-))

Event 2: Arrival of muon at earth

$$t_2 = \frac{d}{\beta c} \quad , \quad x_2 = d$$

$$t_2' = \tau \quad , \quad x_2' = 0$$

L.T. $\quad t_2 = \dfrac{d}{\beta c} = \gamma\left(t_2' + \dfrac{\beta x_2'}{c}\right) = \gamma \tau$

- Observer S explains muon's arrival as being due to time dilation:
  - Time elapsed in muon frame $\tau = t_2/\gamma$.
- Observer S' explains muon's arrival as being due to length contraction:
  - $\tau = \dfrac{d}{\gamma} \cdot \dfrac{1}{\beta c}$

  The distance the surface of the earth has to travel to arrive at the muon is d/γ.

Q5)

(b) How do we reconcile these two <u>correct</u> statements?

(i) From the point of view of observer S' it is the clocks in S that are moving and therefore tick more slowly than clocks in S. When viewed from S', elapsed time on S clocks between Events 1 & 2 is τ/γ!

(ii) It is an objective fact that at Event 2:

Clock in S measures time $t_2 = \dfrac{d}{\beta c}$

Clock in S' measures time $\tau = \dfrac{1}{\gamma}\dfrac{d}{\beta c} = \dfrac{t_2}{\gamma} < t_2$.

Think about this before turning over the page!

Let's define an <u>Event 3</u>

Simultaneously <u>in frame S'</u> with the production of the muon we read the time on the clock (t) at the surface of the earth.

L.T. $0 = t_3' = \gamma\left(t_3 - \frac{\beta x_3}{c}\right)$

$$\therefore \quad t_3 = \frac{\beta d}{c} = \gamma \beta^2 \tau$$

So, the "trap" was to forget that although the clocks in S are synchronised as far as an observer in S is concerned they are <u>not synchronised</u> when viewed from S'.

Therefore time registered by clock at earth when muon arrives is:

$$t_2 = \underbrace{\gamma\beta^2\tau}_{\text{initial time}} + \underbrace{\frac{\tau}{\gamma}}_{\text{elapsed time}} = \gamma\tau\left(\beta^2 + \frac{1}{\gamma^2}\right) = \gamma\tau,$$

thus reconciling (i) and (ii) above!

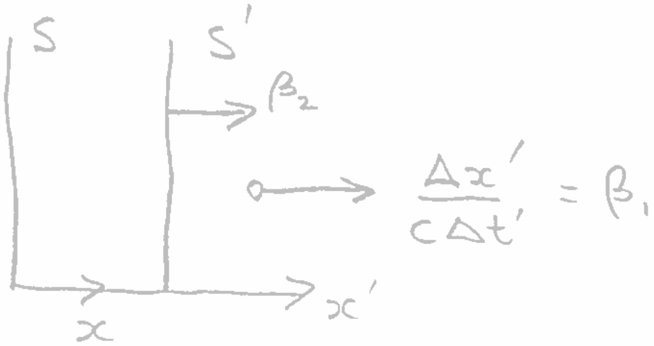
Q5 (c) The time interval in frame S' between Events 1 and 2 is measured <u>with the same clock</u>

$$\therefore \quad \Delta x' = 0$$

Since $\underbrace{(c\Delta t')^2}_{=(c\Delta\tau)^2} - \underbrace{(\Delta x')^2}_{=0} = (c\Delta t)^2 - \underbrace{(\Delta x)^2}_{\geqslant 0}$

$$\Delta t \geqslant \Delta \tau.$$

Q6)



Consider particle moving in S' with $\frac{\Delta x'}{c\Delta t} = \beta_1$

Speed of particle in S :

$$\beta_3 = \frac{\Delta x}{c\Delta t} = \frac{\gamma(\Delta x' + \beta_2 c\Delta t')}{\gamma(c\Delta t' + \beta_2 \Delta x')} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$$