PHYS30441 Electrodynamics: Additional Revision Problems

Note: I've tried to provide a quite large number of revision problems. Don't worry if you don't manage to work through them all within the first three weeks of the semester. That would be a considerable achievement! I do recommend you try to get down at least to question 10, though. You might want to return to this example sheet as part of your revision during reading week, or after the course has finished.

1. (a) Consider the pre-Maxwell equations: Ampere's Law ($\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$) and Gauss' Law. Show that these are *inconsistent* with charge conservation, i.e.

   $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

   (b) Show that if instead the full Maxwell equation for $\nabla \times \mathbf{B}$, including the displacement current, is used together with Gauss' Law, the inconsistency is removed. In other words, we can derive the charge conservation equation from Maxwell's equations.

2. An electrical discharge has an electric field of 10^7 Vm^{-1}. Calculate the energy density.

3. At the centre of the ATLAS experiment at the CERN LHC is a superconducting solenoidal magnet. This magnet is 5.3 m long and has a radius of 1.25 m; it has around 1100 turns, each carrying a current of 7700 A. Using the approximation that this can be considered as an "ideal" or "infinitely long" solenoid, show that the magnetic field produced by the solenoid is around 2.0 T. Calculate the stored magnetic energy, ignoring any fields outside the cylindrical volume of the solenoid.

4. Using Maxwell's equations show that the electric field $\mathbf{E}$ can be written in terms of a vector and scalar potential:

   $$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.$$

5. Starting from Maxwell's equations show that in a region free of charge and current, the electric and magnetic fields in vacuum satisfy the wave equations:

   $$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \qquad \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0.$$

6. Suppose we have a vector potential $\mathbf{A}$ and scalar potential V. Show that replacing these with new potentials $\mathbf{A} \to \mathbf{A} + \nabla g$, $V \to V - \partial g/\partial t$ (where g is any scalar function) gives exactly the same electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. This is called a 'gauge transformation'.

7. Consider the vector potential

   $$\mathbf{A} = B_0\{\frac{yz}{a}\sin\omega t\hat{\mathbf{y}} + (\frac{x^3}{a^2} + 2z)\cos\omega t\hat{\mathbf{z}}\},$$

   where B_0, a and ω are constants.

   (a) Find an expression for a scalar potential V which satisfies the Lorenz gauge condition. Why is this expression for V not unique? (You can choose any suitable solution).

   (b) Calculate the corresponding electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

8. Prove the following results in vector calculus:

   (a) The curl of a gradient is always zero, that is,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} V) = 0,$$

   where V is a scalar field.

   (b) The divergence of a curl is always zero, that is,

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0,$$

   where $\boldsymbol{A}$ is a vector field.

   (c)

$$\boldsymbol{\nabla}(fg) = f(\boldsymbol{\nabla} g) + g(\boldsymbol{\nabla} f),$$

   where f and g are a scalar fields.

   (d)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{f A}) = f(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + (\boldsymbol{\nabla} f) \cdot \boldsymbol{A},$$

   where f is a scalar field and $\boldsymbol{A}$ is a vector field.

   Notes:

   - *These proofs are easiest to establish in cartesian coordinates. (Once proven in cartesian coordinates they must be valid for all coordinate systems.)*
   - *You may wish to write down these proofs in* ***two*** *ways: (i) picking one of the cartesian coordinates, say x, and then writing out explicitly all terms contributing to the x coordinate of the vector expression to be evaluated, and (ii) using general index notation for 3-dimensional vectors.*

9. A charged pion π^- of mass m_π is at rest. It decays to produce a muon μ^- of (rest) mass m_μ and a (massless) neutrino.

   Find, in terms of m_π and m_μ, (a) the energy and (b) the speed of the μ^-.

10. Consider a straight, cylindrical, infinitely long beam of positrons with a charge density per unit volume ρ. The beam has radius a and the positrons travel with speed v in the $\hat{\boldsymbol{z}}$ direction of a cylindrical coordinate system (s, ϕ, z), where s is the distance from the axis of the beam.

    (a) By applying Gauss' law to a cylindrical surface co-axial with the beam, calculate the electric field $\boldsymbol{E}$ both inside and outside the beam ($s < a$ and $s > a$ respectively).

    (b) By applying Ampere's law to a similar surface, calculate the magnetic field $\boldsymbol{B}$ both inside and outside the beam.

    (c) Find expressions for the scalar potential V that are consistent with the expressions you obtained in part (a) for the electric field $\boldsymbol{E}$ for $s < a$ and $s > a$.

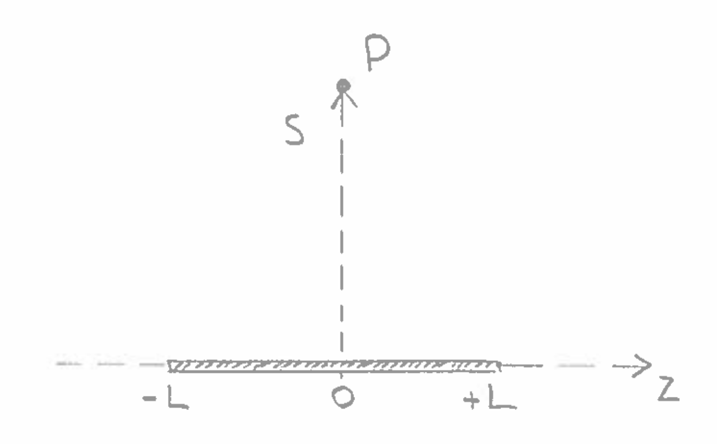
    (d) Find expressions for the vector potential A that are consistent with the expressions you obtained in part (b) for the magnetic field $\boldsymbol{B}$ for $s < a$ and $s > a$.

    (e) Draw a sketch showing the variation of the electric and magnetic fields with s.

    (f) By applying the Lorentz force equation, show that the nett force on a beam particle is radial and find expressions for its magnitude for $s < a$ and $s > a$.

    Note: try to perform as many cross-checks of your results as you can come up with.

11. (a) Consider a straight wire of length $2L$ that carries a line charge density λ, as shown in the figure below.



Using the expression

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\boldsymbol{r'})}{R}\mathrm{d}\ell',$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$, calculate the potential V at the point P a radial distance s from the centre of the wire. Hence, find the electric field, $\boldsymbol{E}$, at the point P.

You may wish to use the standard integral:

$$\int \left[a^2 + x^2\right]^{-\frac{1}{2}}\, dx = \ln\left(\left[a^2 + x^2\right]^{\frac{1}{2}} + x\right).$$

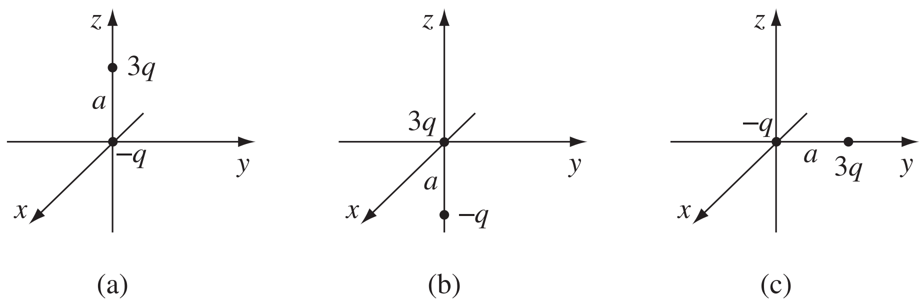
(b) Similarly, consider a straight wire segment of length $2L$ that carries a current I. Using the expression

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int \frac{1}{R}\,\mathbf{d}\boldsymbol{\ell'},$$

calculate the vector potential $\boldsymbol{A}$ a radial distance s from the centre of the wire segment. Hence, find the magnetic field, $\boldsymbol{B}$.

[*Hint:* Cross check your answers for $\boldsymbol{E}$ and $\boldsymbol{B}$ by considering the case $L \to \infty$ and comparing with your answers to question 10 (a) and (b) above.]

12. Two point charges, $3q$ and $-q$ are separated by a distance a. For each of the arrangements given in the following figure:



(i) Find the dipole moment.

(ii) Find an approximate expression for the potential, V, a large distance r from the charges. Work in spherical polar coordinates and include both the monopole and dipole contributions to V.

13. This question contains various pieces of "bookwork" relating to electric dipoles.

(a) In Lecture 5 we derived an expression for the electric dipole moment of a charge distribution:

$$\boldsymbol{p} = \int_{\mathcal{V}} \boldsymbol{r'} \rho(\boldsymbol{r'})\, \mathrm{d}\tau'.$$

If the origin of the coordinate system is displaced by $\boldsymbol{a}$ (a vector displacement) find an expression for the new dipole moment $\boldsymbol{p'}$. Thus, show that if the total charge is zero, the electric dipole moment is unchanged by a displacement.

(b) In Lecture 5 we derived also an expression for the potential for an electric dipole:

$$V_{\text{dipole}}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{\widehat{\boldsymbol{r}} \cdot \boldsymbol{p}}{r^2}. \tag{1}$$

i. If we choose coordinates such that the dipole is at the origin and points along the z direction, show that:

$$\boldsymbol{E}_{\text{dipole}} = \frac{p}{4\pi\epsilon_0 r^3} \left(2\cos\theta\, \widehat{\boldsymbol{r}} + \sin\theta\, \widehat{\boldsymbol{\theta}}\right).$$

Note the similarity between this expression and the one we obtained in Lecture 7 for the field of a magnetic dipole $\boldsymbol{m} = m\widehat{\boldsymbol{z}}$ at the origin.

ii. Verify that the "coordinate-free" form for the electric field of an electric dipole:

$$\boldsymbol{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0 r^3} \left(3\left[\boldsymbol{p} \cdot \widehat{\boldsymbol{r}}\right] \widehat{\boldsymbol{r}} - \boldsymbol{p}\right), \tag{2}$$

is consistent with the result obtained in part (i) above.

iii. *(More Difficult) Prove* the "coordinate-free" form of Equation 2 by applying $\boldsymbol{E} = -\boldsymbol{\nabla} V$ to Equation 1, for an electric dipole of arbitrary direction $\boldsymbol{p} = p\widehat{\boldsymbol{p}}$.

Suggested method:

A. Employ vector identities (3) and (4) to expand out $\boldsymbol{\nabla} V$.

B. Show that three of the resulting terms in $\boldsymbol{\nabla} V$ are zero.

C. Work out explicitly in cartesian coordinates the remaining two (non-zero) terms.

Note the difference between "Verify" and "Prove" in parts ii. *and* iii. *above.*

14. This question contains a couple of pieces of "bookwork" relating to magnetic dipoles.

(a) Use Stokes' Theorem on the vector field $\mathbf{v} = \phi\mathbf{c}$, where ϕ is a scalar field and $\mathbf{c}$ is a constant vector, to show that:

$$-\int_S \boldsymbol{\nabla}\phi \times \mathrm{d}\mathbf{a} = \oint_C \phi\, \mathrm{d}\mathbf{l},$$

where the surface S is bounded by the contour C.

[*Hint:* You may find it useful to use the vector identity

$$\boldsymbol{\nabla} \times (\phi\mathbf{u}) = \phi(\boldsymbol{\nabla} \times \mathbf{u}) - \mathbf{u} \times (\boldsymbol{\nabla}\phi).\ ]$$

Re-writing the above result in terms of the primed coordinates

$$-\int_S \boldsymbol{\nabla}_{r'}\, \phi \times \mathrm{d}\mathbf{a}' = \oint_C \phi\, \mathrm{d}\mathbf{l}',$$

and by considering $\phi = \hat{\mathbf{r}} \cdot \mathbf{r}'$, obtain the result used in the discussion of magnetic dipole moments (Lecture 7), namely:

$$\oint_C (\hat{\mathbf{r}} \cdot \mathbf{r}')\, \mathrm{d}\mathbf{l}' = -\hat{\mathbf{r}} \times \int_S \mathrm{d}\mathbf{a}' = -\hat{\mathbf{r}} \times \mathbf{a}.$$

The quantity $\mathbf{a}$ is known as the *vector area* of the loop. If the loop is flat, then this is equal in magnitude to the area of the loop with a direction given (with respect to the current in the loop) by the Right-Hand Rule.

(b) Verify that the "coordinate-free" form for the magnetic field of a magnetic dipole:

$$\boldsymbol{B}_{\text{dipole}} = \frac{\mu_0}{4\pi r^3} \left(3\left[\boldsymbol{m} \cdot \widehat{\boldsymbol{r}}\right] \widehat{\boldsymbol{r}} - \boldsymbol{m}\right),$$

is consistent with the expression we obtained in Lecture 7 for the field of a magnetic dipole $\boldsymbol{m} = m\widehat{\boldsymbol{z}}$ at the origin.