PHYS30441 Electrodynamics: Additional Revision Problems - Solutions

1. (a) Start with Gauss' Law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. Take the time derivative
$$\Rightarrow \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_0}\frac{\partial \rho}{\partial t}$$
Substituting from charge conservation
$$\Rightarrow \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{\epsilon_0}\nabla \cdot \mathbf{j}.$$
But Ampere's Law gives
$$\nabla \cdot \mathbf{j} = \frac{1}{\mu_0}\nabla \cdot \nabla \times \mathbf{B} = 0.$$
This leads to a contradiction since $\nabla \cdot \frac{\partial \mathbf{E}}{\partial t}$ is not always zero.

   (b) Taking the divergence of the generalized Ampere's Law:
$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right).$$
The identity $\nabla \cdot (\nabla \times \mathbf{B}) = 0$ and Gauss' Law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ allows the above to be written as
$$0 = \nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t}.$$
This is the continuity of charge.

2. Energy density
$$\frac{1}{2}\epsilon_0 E^2 \simeq 440 \text{ Jm}^{-3}.$$

3. Field
$$B \simeq \frac{\mu_0 N I}{L} \simeq \frac{4\pi \times 10^{-7} \times 1100 \times 7700}{5.3} \simeq 2.0 \text{ T}.$$
Stored energy
$$U_m = \frac{1}{2\mu_0}\int_V B^2 dV \simeq \frac{1}{2\mu_0}B^2 \times \pi r^2 L \simeq \frac{1}{2 \times 4\pi \times 10^{-7}} \times 2^2 \times \pi \times (1.25)^2 \times 5.3 \simeq 4.1 \times 10^7 \text{ J}.$$

4. Consider the vector identity $\nabla \times (\nabla V) = 0$. The converse also holds: if $\nabla \times \mathbf{v} = 0$, we can find V such that $\mathbf{v} = \nabla V$. Similarly, considering the vector identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, the converse also holds: if $\nabla \cdot \mathbf{B} = 0$, we can find $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$. (*Note that the latter works for static and for time-varying fields* $\mathbf{B}$.)

   From Faraday's Law
$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \quad \Longrightarrow \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$
Hence we can write
$$\mathbf{E} + \frac{\partial A}{\partial t} = -\nabla V.$$
Thus we can represent the EM fields as
$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

5. Starting with $\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\nabla \times \mathbf{B}$ and using

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t} \text{ with } \mathbf{j} = \mathbf{0} \text{ and } \epsilon_0\mu_0 = \frac{1}{c^2}$$

$$\Rightarrow \nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right).$$

The standard vector identity allows the left-hand side to be expanded:

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2(E) = -\nabla^2(E)$$

where Gauss' Law for a region free of sources has been applied, hence $\nabla \cdot \mathbf{E} = 0$.

Thus we have:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

A similar approach taking the curl of the generalised Ampere's equation should be followed to also obtain

$$\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0.$$

6. If $\mathbf{A}$ is a vector potential for $\mathbf{B}$, then $\nabla \times \mathbf{A} = \mathbf{B}$. Hence

$$\nabla \times (\mathbf{A} + \nabla g) = \nabla \times \mathbf{A} + \nabla \times \nabla g = \nabla \times \mathbf{A} = \mathbf{B}.$$

(since $\nabla \times \nabla g = 0$; a vector identity). Thus $\mathbf{A} + \nabla g$ is also a vector potential for $\mathbf{B}$.

Similarly, using

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t},$$

we obtain

$$\begin{aligned} -\nabla\left(V - \frac{\partial g}{\partial t}\right) - \frac{\partial}{\partial t}(\mathbf{A} + \nabla g) &= -\nabla V + \nabla\frac{\partial g}{\partial t} - \frac{\partial}{\partial t}\nabla g - \frac{\partial \mathbf{A}}{\partial t} \\ &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}. \end{aligned}$$

So,

$$\mathbf{A} \to \mathbf{A} + \nabla g, V \to V - \frac{\partial g}{\partial t}$$

are also valid potentials for $\mathbf{E}$.

7. The Lorenz gauge condition is

$$\frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla \cdot \mathbf{A} = 0,$$

where

$$\frac{1}{c^2} = \mu_o\epsilon_0.$$

Hence

$$\frac{\partial V}{\partial t} = -c^2 B_0 \nabla \cdot \left[\frac{yz}{a}\sin\omega t\hat{\mathbf{y}} + \left(\frac{x^3}{a^2} + 2z\right)\cos\omega t\hat{\mathbf{z}}\right] = -c^2 B_0\left(\frac{z}{a}\sin\omega t + 2\cos\omega t\right).$$

$$\text{Integrating w.r.t. } t \Rightarrow V = c^2\frac{B_0}{\omega}\left(\frac{z}{a}\cos\omega t - 2\sin\omega t\right) + V_0(x, y, z).$$

Note that the 'constant of integration' is an arbitrary function of space (x, y, z) so the scalar potential V is NOT unique. We can add any spatial function, but the obvious (and simplest) choice is $V_0 = 0$. Proceeding with this,

$$\begin{aligned}
B &= \nabla \times \mathbf{A} \\
&= B_0 \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \frac{yz}{a}\sin\omega t & \left(\frac{x^3}{a^2} + 2z\right)\cos\omega t \end{vmatrix} \\
&= B_0 \left(-\hat{\mathbf{x}}\frac{y}{a}\sin\omega t - \hat{\mathbf{y}}\frac{3x^2}{a^2}\cos\omega t \right).
\end{aligned}$$

(This is unique, for the given vector potential). Also,

$$\begin{aligned}
\mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\
&= -\frac{c^2 B_0}{a\omega}\cos\omega t\hat{\mathbf{z}} - B_0\frac{yz}{a}\omega\cos\omega t\hat{\mathbf{y}} + B_0\left(\frac{x^3}{a^2} + 2z\right)\omega\sin\omega t\hat{\mathbf{z}} \\
&= B_0\left[-\frac{c^2}{a\omega}\cos\omega t\hat{\mathbf{z}} - \frac{\omega}{a}yz\cos\omega t\hat{\mathbf{y}} + \left(\frac{x^3}{a^2} + 2z\right)\omega\sin\omega t\hat{\mathbf{z}}\right]
\end{aligned}$$

(You may obtain a different answer if you chose a different V — differing by ∇V_0).

Q8 (a)

$$\nabla \times (\nabla V) = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= \hat{\underline{x}} \underbrace{\left[\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial y}\right) \right]}_{=0}$$

and similarly for the $\hat{\underline{y}}$ and $\hat{\underline{z}}$ terms.

Or, alternatively, in index notation

$$\nabla \times (\nabla V) = \varepsilon_{ijk}\, \hat{\underline{x}}_i \frac{\partial}{\partial x_j}\left(\frac{\partial V}{\partial x_k}\right)$$

With respect to the interchange $j \leftrightarrow k$:

ε_{ijk} is antisymmetric

$\frac{\partial}{\partial x_j}\left(\frac{\partial V}{\partial x_k}\right)$ is symmetric

$\therefore$ Each component i is zero.

Q8(b)

$$\nabla . (\nabla \times \underline{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 0 ,$$

since $\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x}\right)$, etc

Index notation

$$\nabla . (\nabla \times \underline{A}) = \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk} \frac{\partial A_k}{\partial x_j}\right) = \varepsilon_{ijk} \frac{\partial}{\partial x_i}\left(\frac{\partial A_k}{\partial x_j}\right) = 0 ,$$

since with respect to interchange $i \leftrightarrow j$

ε_{ijk} antisymmetric

$\frac{\partial}{\partial x_i}\left(\frac{\partial A_k}{\partial x_j}\right)$ symmetric.

Q8 (c)

Let's consider the $\hat{\underline{x}}$ component of $\nabla(fg)$.

$$\left[\nabla(fg)\right]_x = \frac{\partial}{\partial x}(fg) = f\left(\frac{\partial g}{\partial x}\right) + g\left(\frac{\partial f}{\partial x}\right)$$

$$= f\left[\nabla g\right]_x + g\left[\nabla f\right]_x$$

Same results hold for the $\hat{\underline{y}}$ and $\hat{\underline{z}}$ components

$$\therefore \quad \nabla(fg) = f(\nabla g) + g(\nabla f)$$

In index notation:

$$\nabla(fg) = \hat{\underline{x}}_i \frac{\partial}{\partial x_i}(fg) = f\left(\hat{\underline{x}}_i \frac{\partial g}{\partial x_i}\right) + g\left(\hat{\underline{x}}_i \frac{\partial f}{\partial x_i}\right)$$

$$= f(\nabla g) + g(\nabla f)$$

Q8 (d)

$$\nabla.\left(f\underline{A}\right) = \frac{\partial}{\partial x}\left(fA_x\right) + \frac{\partial}{\partial y}\left(fA_y\right) + \frac{\partial}{\partial z}\left(fA_z\right)$$

$$= f\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + \left(A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z}\right)$$

$$= f\left(\nabla.\underline{A}\right) + \underline{A}.\left(\nabla f\right)$$

In index notation

$$\nabla.\left(f\underline{A}\right) = \frac{\partial}{\partial x_i}\left(fA_i\right) = f\left(\frac{\partial A_i}{\partial x_i}\right) + A_i\left(\frac{\partial f}{\partial x_i}\right)$$

$$= f\left(\nabla.\underline{A}\right) + \underline{A}.\left(\nabla f\right)$$

Q9) $\mu^- \longleftarrow \underset{\pi^-}{\bullet} \longrightarrow \bar{\nu}_\mu$

4-momenta: $\left(\frac{E_\mu}{c}, \underline{p}\right)$ $\quad (m_\pi c, 0)$ $\quad (p, -\underline{p})$

because ν is massless

to ensure conservation of momentum in decay.

(a) $E_\mu^2 = p^2c^2 + (m_\mu c^2)^2 \Rightarrow p^2 = \frac{E_\mu^2}{c^2} - m_\mu^2 c^2$

From conservation of energy:

$$\frac{E_\mu}{c} + p = m_\pi c \Rightarrow p^2 = \left(m_\pi c - \frac{E_\mu}{c}\right)^2$$

$$= m_\pi^2 c^2 - 2m_\pi E_\mu + \frac{E_\mu^2}{c^2}$$

Equating the right-hand sides gives

$$E_\mu = \frac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}$$

Q9 (b)

From $E_\mu = \gamma m_\mu c^2$ and answer to part (a)

$$\gamma = \frac{m_\pi^2 + m_\mu^2}{2 m_\mu m_\pi}$$

From $\gamma^2 = (1 - \beta^2)^{-1}$

$$\beta^2 = 1 - \gamma^{-2} = 1 - \frac{4 m_\mu^2 m_\pi^2}{(m_\pi^2 + m_\mu^2)^2}$$

$$= \frac{(m_\pi^2 + m_\mu^2)^2 - 4 m_\mu^2 m_\pi^2}{(m_\pi^2 + m_\mu^2)^2}$$

$$= \frac{(m_\pi^2 - m_\mu^2)^2}{(m_\pi^2 + m_\mu^2)^2}$$

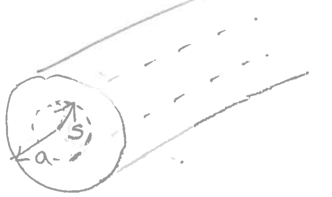
$$\therefore \quad \beta = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}$$ (speed in units of c)

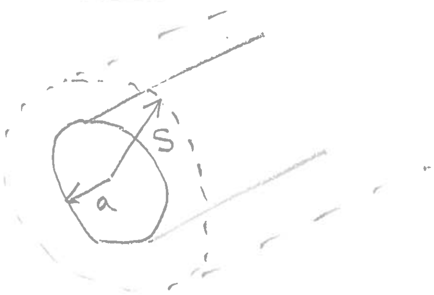
or $v = \beta c$

Question 10)

Within beam ($s<a$) | **Outside beam ($s>a$)**

(a)





Charge per unit length enclosed by surface

$\lambda = \pi s^2 \rho$ $\qquad\qquad$ $\lambda = \pi a^2 \rho$

Flux of E out of surface per unit length

$\Phi = 2\pi s E$ $\quad$ {By symmetry E is radially outwards as ρ is positive}

Apply Gauss' Law

$2\pi s E = \pi s^2 \rho / \varepsilon_0$ $\qquad\qquad$ $2\pi s E = \pi a^2 \rho / \varepsilon_0$

$$\underline{E} = \frac{s\rho}{2\varepsilon_0}\hat{s} \qquad\qquad \underline{E} = \frac{a^2\rho}{2\varepsilon_0 s}\hat{s}$$

Cross check $\quad$ at $s=a$ both expressions give

$$E = \frac{a\rho}{2\varepsilon_0}$$

N.B. In the question you are asked for the (vector) $\underline{E}$. A complete answer therefore requires the direction as well as the magnitude!

Q10(b) **Within beam** | **Outside beam**

Beam current through surface ⊥ beam

Within beam:

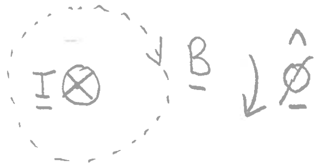
$$\underline{I} = \pi s^2 \rho v \,\hat{\underline{z}}$$

Outside beam:

$$\underline{I} = \pi a^2 \rho v \,\hat{\underline{z}}$$

$\oint B\,dl$ around surface (By symmetry B is circumferential)

$$\oint B\,dl = 2\pi s B$$



Apply Ampere's Law

Within beam:

$$2\pi s B = \mu_0 I = \mu_0 \pi s^2 \rho v$$

$$\underline{B} = \frac{\mu_0 \rho v s}{2}\,\hat{\underline{\phi}}$$

Outside beam:

$$2\pi s B = \mu_0 \pi a^2 \rho v$$

$$\underline{B} = \frac{\mu_0 a^2 \rho v}{2 s}\,\hat{\underline{\phi}}$$

Cross check at $s = a$ both expressions give

$$B = \frac{\mu_0 \rho v a}{2}$$

N.B. Sign of $\underline{B}$ relative to $\underline{I}$, $\hat{\underline{\phi}}$ from right hand rule.

Q10(c)

Within beam

$s < a$

$$\underline{E} = \frac{s\rho}{2\varepsilon_0}\hat{\underline{s}} = -\nabla V = -\frac{\partial V}{\partial s}\hat{\underline{s}}$$

$$\therefore \quad V = -\frac{s^2\rho}{4\varepsilon_0}$$

choosing $V=0$ at $s=0$

Outside beam

$s > a$

$$\underline{E} = \frac{a^2\rho}{2\varepsilon_0 s}\hat{\underline{s}}$$

$$V = -\frac{a^2\rho}{2\varepsilon_0}\ln s + K$$

Ensuring continuity at $s=a$

$$V = -\frac{a^2\rho}{2\varepsilon_0}\left\{\ln\left(\frac{s}{a}\right) + \frac{1}{2}\right\}$$

(d)

Within beam

$$\underline{B} = \frac{\mu_0\rho v s}{2}\hat{\underline{\phi}}$$

$$= \nabla \times \underline{A} = -\frac{\partial A_z}{\partial s}\hat{\underline{\phi}}$$

$\underline{A}$ is in $\hat{\underline{z}}$ direction as is $\underline{I}$

(Using definition of curl in cylindrical coordinates and $\underline{A} = A_z\hat{\underline{z}}$, as well as symmetry in $\phi \Rightarrow \frac{\partial A_z}{\partial \phi} = 0$!)

$$\therefore \quad \underline{A} = -\frac{\mu_0\rho v s^2}{4}\hat{\underline{z}}$$

choosing $A=0$ at $s=0$

Outside beam

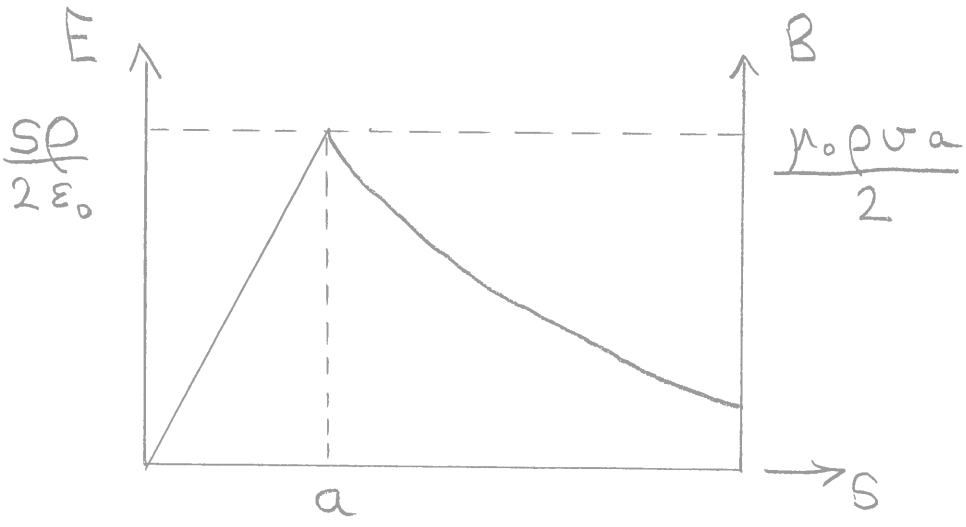
$$\underline{B} = \frac{\mu_0 a^2\rho v}{2s}\hat{\underline{\phi}}$$

$$\underline{A} = -\frac{\mu_0 a^2\rho v}{2}\ln s + K$$

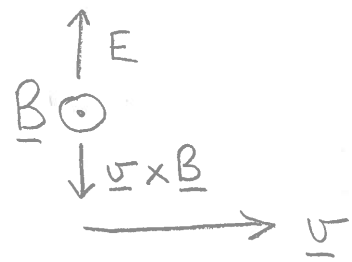
Ensuring continuity at $s=a$

$$A = -\frac{\mu_0 a^2\rho v}{2}\left\{\ln\frac{s}{a} + \frac{1}{2}\right\}$$

Q10(e)



(f)



Magnetic and electrical forces are in opposite directions

$$\therefore \underline{F} = e\left(\underline{E} + \underline{v} \times \underline{B}\right)$$ is radial

Within beam	Outside beam
$F = e\left(\frac{s\rho}{2\varepsilon_0} - \frac{v^2\mu_0 \rho s}{2}\right)$	$F = e\left(\frac{a^2\rho}{2\varepsilon_0 s} - \frac{v^2\mu_0 a^2 \rho}{2s}\right)$
$= \frac{es\rho}{2}\left(\frac{1}{\varepsilon_0} - v^2\mu_0\right)$	$= \frac{ea^2\rho}{2s}\left(\frac{1}{\varepsilon_0} - v^2\mu_0\right)$

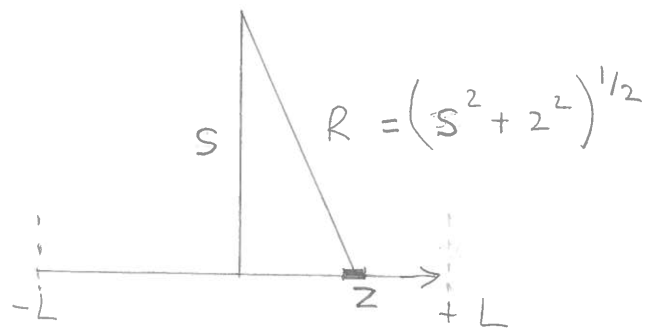
In both cases it is tempting to surmise that

$$F \to 0 \quad \text{as} \quad v^2 \to \frac{1}{\varepsilon_0\mu_0} = c^2$$

However, we shall return to this problem after we have derived the relativistic equations for $\rho, \underline{j}, V, \underline{A}, \underline{E}, \underline{B}$ etc.

Question 11)

(a)



$$V = \frac{\lambda}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{dz}{(s^2+z^2)^{1/2}} = \frac{2\lambda}{4\pi\varepsilon_0} \int_{0}^{L} \frac{dz}{(s^2+z^2)^{1/2}}$$

(by symmetry)

$$= \frac{2\lambda}{4\pi\varepsilon_0} \left[\ln\left\{ (s^2+z^2)^{1/2} + z \right\} \right]_0^L$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \left[\ln\left\{ (L^2+s^2)^{1/2} + L \right\} - \ln s \right]$$

$$E = -\frac{\partial V}{\partial s} = \frac{-\lambda}{2\pi\varepsilon_0} \left[\frac{s(L^2+s^2)^{-1/2}}{(L^2+s^2)^{1/2}+L} - \frac{1}{s} \right]$$

As $L \to \infty$ $\quad E \to \dfrac{\lambda}{2\pi\varepsilon_0 s}$ $\quad$ (Field is radially outwards if λ is positive)

This is consistent with the (outside beam) result in Q10(a) when $\lambda = \pi a^2 \rho$.

Alternative approach to definite integral in Q11) (a)

$$V = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left\{(s^2+z^2)^{1/2}+z\right\}\right]_{-L}^{L}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left\{(s^2+L^2)^{1/2}+L\right\} - \ln\left\{(s^2+L^2)^{1/2}-L\right\}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left\{\frac{(s^2+L^2)^{1/2}+L}{(s^2+L^2)^{1/2}-L}\right\}$$

multiply top and bottom by $(s^2+L^2)^{1/2}+L$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left\{\frac{\left[(s^2+L^2)^{1/2}+L\right]^2}{(s^2+\cancel{L^2})-\cancel{L^2}}\right\}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\,2\left[\ln\left\{\frac{(s^2+L^2)^{1/2}+L}{s}\right\}\right]$$

which is equivalent to the result obtained using $2\left[\ \ \right]_0^L$.

Q11 (b) Since $\underline{I} = I\hat{z}$ has only one component the solution for $\underline{A} = A\hat{z}$ is very similar to that for V in part (a).

$$A = \frac{2\mu_0 I}{4\pi}\int_0^L \frac{dz}{(s^2+z^2)^{1/2}}$$

$$= \frac{\mu_0 I}{2\pi}\left[\ln\left\{(L^2+s^2)^{1/2}+L\right\} - \ln s\right]$$

Since A_z is the only non-zero component and it varies only as a function of s then in the expression for $\nabla \times \underline{A}$ in cylindrical coordinates

$$\underline{B} = -\frac{\partial A_z}{\partial s}\hat{\underline{\phi}} = -\frac{\mu_0 I}{2\pi}\left[\frac{s(L^2+s^2)^{-1/2}}{(L^2+s^2)^{1/2}+L} - \frac{1}{s}\right]\hat{\underline{\phi}}$$

As $L \to \infty$ $\quad B_\phi \to \dfrac{\mu_0 I}{2\pi s}$

This is consistent with the (outside beam) result in Q10(b) when $I = \pi a^2 \rho v$.

Question 12)

In all three cases the monopole term is

$$V_0(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}(3q - q) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{r}$$

(i) The dipole moment $\underline{p} = \int \underline{r}'\rho(\underline{r}')\,d\tau'$

(a) $\underline{p} = 3qa\,\hat{\underline{z}}$

(b) $\underline{p} = (-q)(-a)\hat{\underline{z}} = qa\,\hat{\underline{z}}$

(c) $\underline{p} = 3qa\,\hat{\underline{y}}$

$$V_{dipole} = \frac{\hat{\underline{r}}\cdot\underline{p}}{4\pi\varepsilon_0 r^2} \qquad \hat{\underline{r}} = \sin\theta\cos\phi\,\hat{\underline{x}} + \sin\theta\sin\phi\,\hat{\underline{y}} + \cos\theta\,\hat{\underline{z}}$$

(ii) (a) $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2} + \cdots\cdots\right]$

(b) $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2} + \cdots\cdots\right]$

(c) $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2} + \cdots\cdots\right]$

Q13)

(a) We want to keep the same charge distribution $\rho(\underline{r}')$, but we want its location with respect to the origin to be $\underline{r}' - \underline{a}$.

$$\therefore \quad \underline{p} \to \underline{p}' = \int_{v} (\underline{r}' - \underline{a}) \rho(\underline{r}') \, d\tau'$$

$$= \int_{v} \underline{r}' \rho(\underline{r}') \, d\tau' - \underline{a} \int_{v} \rho(\underline{r}') \, d\tau'$$

can be taken outside the integral because it is a constant.

$$= \underline{p} - \underline{a} Q$$, where Q is the total charge.

$\therefore$ $\underline{p}$ is unchanged by a displacement if the total charge is zero.

Q13)
(b)(i)

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{\hat{\underline{r}} \cdot \underline{p}}{r^2},$$

with $\underline{p} = p\,\hat{\underline{z}}$ as required by question

$$\therefore \quad V_{dipole} = \frac{p}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2} \quad . \quad \underline{E} = -\nabla V.$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p}{4\pi\varepsilon_0} \frac{\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

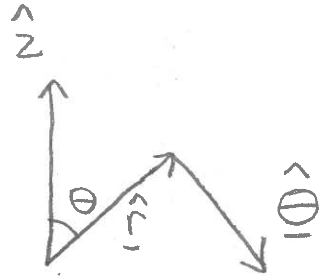
$$E_\phi = \frac{-1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

$$\therefore \quad \underline{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)$$

Q13)

(b) (ii) For the dipole in part (i):

$$\underline{p} = p\hat{\underline{z}} = p\left(\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}\right)$$



$$\therefore\ \left[\underline{p}\cdot\hat{\underline{r}}\right]\hat{\underline{r}} = \left[p\cos\theta\right]\hat{\underline{r}}$$

$$\frac{1}{4\pi\varepsilon_0 r^3}\left(3\left[\underline{p}\cdot\hat{\underline{r}}\right]\hat{\underline{r}} - \underline{p}\right) = \frac{1}{4\pi\varepsilon_0 r^3}\left(3p\cos\theta\,\hat{\underline{r}} - p\left[\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}\right]\right)$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right),$$

which is consistent with the result in part (i).

Q13)

(b)(iii) $\underline{E}_{dipole} = -\nabla V = \frac{-1}{4\pi\varepsilon_0} \nabla\left(\frac{\hat{\underline{r}} \cdot \hat{\underline{p}}}{r^2}\right)$ (Eqn 1.1)

Using vector identity (3)

$$\nabla\left(\frac{\hat{\underline{r}} \cdot \underline{p}}{r^2}\right) = \left(\hat{\underline{r}} \cdot \underline{p}\right)\nabla\left(\frac{1}{r^2}\right) + \frac{1}{r^2}\nabla\left(\hat{\underline{r}} \cdot \underline{p}\right)$$

$$= \left(\hat{\underline{r}} \cdot \underline{p}\right)\nabla\left(\frac{1}{r^2}\right) + \frac{1}{r^2}\left[\hat{\underline{r}} \times (\nabla \times \underline{p}) + \underline{p} \times (\nabla \times \hat{\underline{r}}) + (\hat{\underline{r}} \cdot \nabla)\hat{\underline{p}} + (\underline{p} \cdot \nabla)\hat{\underline{r}}\right]$$

(Eqn 1.2)

[using vector identity (4)]

This looks horendous, but bear with me!

$\underline{p} = p\hat{\underline{p}}$ where p is a constant scalar and $\hat{\underline{p}}$ is a constant vector.

$\therefore$ $\nabla \times \underline{p}$ and $(\hat{\underline{r}} \cdot \nabla)\underline{p}$ are trivially zero.

Also since $\hat{\underline{r}}$ is a radial vector $\nabla \times \hat{\underline{r}}$ must be zero, but let's show this for one component explicitly:

$$\left[\nabla \times \hat{\underline{r}}\right]_x = \left[\nabla \times \left(\frac{\underline{r}}{r}\right)\right]_x = \frac{\partial}{\partial y}\left(\frac{z}{r}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r}\right) = z\left(\frac{-1}{2}\frac{2y}{r^3}\right) - y\left(\frac{-1}{2}\frac{2z}{r^3}\right)$$

$$= 0$$

Thus Equation $(1 \cdot 2)$ becomes

$$\nabla\left(\frac{\hat{\underline{r}} \cdot \underline{p}}{r^2}\right) = \left(\hat{\underline{r}} \cdot \underline{p}\right) \nabla\left(\frac{1}{r^2}\right) + \frac{1}{r^2}\left(\underline{p} \cdot \nabla\right)\hat{\underline{r}} \qquad (\text{Eqn } 1 \cdot 3)$$

Writing out explicitly in cartesian coordinates:

$$\nabla\left(\frac{1}{r^2}\right) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left[x^2+y^2+z^2\right]^{-1}$$

$$= \frac{-2\left(x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}\right)}{\left[x^2+y^2+z^2\right]^{2}} = \frac{-2\underline{r}}{r^4} = \frac{-2\hat{\underline{r}}}{r^3} \qquad (\text{Eqn } 1 \cdot 4)$$

$$\left(\underline{p} \cdot \nabla\right)\hat{\underline{r}} = \left(\underline{p} \cdot \nabla\right)\left(\frac{\underline{r}}{r}\right) = \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{\left(x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}\right)}{\left[x^2+y^2+z^2\right]^{1/2}}$$

$$= \frac{\left(p_x\hat{\underline{x}} + p_y\hat{\underline{y}} + p_z\hat{\underline{z}}\right)}{\left[x^2+y^2+z^2\right]^{1/2}} - \frac{1}{2} \cdot 2\,\frac{\left(p_x x + p_y y + p_z z\right)\underline{r}}{\left[x^2+y^2+z^2\right]^{3/2}}$$

$$= \frac{\underline{p}}{r} - \left(\frac{\underline{r} \cdot \underline{p}}{r^3}\right)\underline{r} \qquad (\text{Eqn } 1 \cdot 5)$$

Substituting Eqns $(1 \cdot 4)$ and $(1 \cdot 5)$ into Eqn $(1 \cdot 3)$ gives

$$\nabla\left(\frac{\hat{\underline{r}} \cdot \underline{p}}{r^2}\right) = -\frac{2}{r^3}\left(\hat{\underline{r}} \cdot \underline{p}\right)\hat{\underline{r}} + \frac{1}{r^2}\left[\frac{\underline{p}}{r} - \left(\frac{\underline{r} \cdot \underline{p}}{r^3}\right)\underline{r}\right]$$

$$= -\frac{3}{r^3}\left(\hat{\underline{r}} \cdot \underline{p}\right)\hat{\underline{r}} + \frac{\underline{p}}{r^3}$$

Substituting into Eqn $(1 \cdot 1)$ yields the desired result:

$$\underline{E}_{\text{dipole}} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\left[\underline{p} \cdot \hat{\underline{r}}\right]\hat{\underline{r}} - \underline{p}\right)$$

Notes:

An alternative treatment of $\nabla\left(\frac{1}{r^2}\right)$ using vector identity (3)

$$\nabla\left(\frac{1}{r^2}\right) = 2\left(\frac{1}{r}\right)\nabla\left(\frac{1}{r}\right) = \frac{2}{r}\left(-\frac{\hat{r}}{r^2}\right) = -\frac{2\hat{r}}{r^3}$$

avoids having to write out cartesian coordinates

Q14) (a)

Stoke's Theorem appled to $\underline{v} = \phi \underline{c}$

$$\int_S (\nabla \times [\phi \underline{c}]) . \underline{da} = \oint_C \phi \underline{c} . \underline{dl} = \underline{c} . \oint_C \phi \, \underline{dl} \quad (A)$$

(since $\underline{c}$ is a constant)

From vector identity (7)

$$\nabla \times [\phi \underline{c}] = \phi \underbrace{(\nabla \times \underline{c})}_{= 0 \text{ since } \underline{c} \text{ is a constant}} - \underline{c} \times [\nabla \phi] \quad (B)$$

Substituting (B) into (A)

$$\underline{c} . \oint_C \phi \, \underline{dl} = - \int_S (\underline{c} \times [\nabla \phi]) . \underline{da} = - \underbrace{\underline{c} . \int [\nabla \phi] \times \underline{da}}_{\text{from vector identity (1)}}$$

Since $\underline{c}$ is an arbitrary constant vector it must be the case that

$$- \int [\nabla \phi] \times \underline{da} = \oint_C \phi \, \underline{dl} \quad (C)$$

as required.

Let $\phi = \hat{\underline{r}} \cdot \underline{r}'$

and re-write eqn. (C) with primed coordinates used for differentials.

$$\oint_c (\hat{\underline{r}} \cdot \underline{r}')\, \underline{dl}' = -\int [\nabla_{r'} (\hat{\underline{r}} \cdot \underline{r}')] \times \underline{da}$$

$$= \int \hat{\underline{r}} \times \underline{da} = \hat{\underline{r}} \times \underline{a} \qquad \text{as required}$$

$$\left[\text{Since } \left(\hat{\underline{x}} \frac{\partial}{\partial x'} + \hat{\underline{y}} \frac{\partial}{\partial y'} + \hat{\underline{z}} \frac{\partial}{\partial z'} \right) \left(\frac{x x' + y y' + z z'}{r} \right) = \frac{\hat{\underline{x}} x + \hat{\underline{y}} y + \hat{\underline{z}} z}{r} = \hat{\underline{r}} \right]$$

Q14) (b)

$$\underline{m} = m\hat{\underline{z}} = m\left(\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}\right)$$

Substituting into the expression given in the question:

$$\therefore \quad \underline{B}_{dipole} = \frac{\mu_0}{4\pi r^3}\left[3m\cos\theta\,\hat{\underline{r}} - m\left(\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}\right)\right.$$

$$= \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right],$$

which is the expression we obtained in Lecture 7.