Electrodynamics (PHYS30441) Additional Examples 2

1. In an inertial frame of reference S' a (massless) photon has a momentum $\boldsymbol{p'}$ that makes an angle θ' with the x^1 axis. Frame S' is moving with respect to frame S in the x^1 direction with speed β (in units of c).

   (a) Using a Lorentz transformation, find the 4-momentum components $(p, \boldsymbol{p})$ of the photon in the frame S. The momentum $\boldsymbol{p}$ should be given in terms of the components parallel and perpendicular to the x^1 direction.

   (b) Hence, find the angle θ between the x^1 axis and the momentum $\boldsymbol{p}$ of the photon in the frame S.

   (c) What relation does this expression have to the formula for adding velocities in special relativity?

   (d) Cross check your answer by verifying that the photon has zero mass in frame S, i.e., that magnitude of the 4-momentum in S is zero.

2. A particle is at rest in frame S' and decays to two photons. In S' the two photons travel with opposite momenta that make an angle θ' with the x^1 axis and have magnitude p'. Let's assume the photon momenta lie in the x^1-x^2 plane. Frame S' is moving with respect to frame S in the x^1 direction with speed β (in units of c).

   (a) Draw diagrams showing qualitatively the 3-momenta of the two photons in frames S' and S.

   (b) Write down the components of the 4-momenta of the two photons in frame S'. I suggest labelling these 4-momenta and their components with a sub-script "+" for the photon whose momentum component along the x^1 direction in frame S' is positive ($\boldsymbol{p'_+}$) and with a sub-script "−" for the photon whose momentum component along the x^1 direction in frame S' is negative ($\boldsymbol{p'_-}$).

   (c) Using a Lorentz transformation, find the 4-momentum components of the two photons $(p_+, \boldsymbol{p}_+)$ and $(p_-, \boldsymbol{p}_-)$ in the frame S.

   (d) Hence, find an expression for the angle between the two photons in the frame S.

3. Show that the flux of $\boldsymbol{B}$ through a surface is given by the integral of $\boldsymbol{A}$ around the line that encloses the surface. That is:

$$\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}.$$

4. The potential $V_0(\theta) = k \sin^2(\theta/2)$ is specified on the surface of a hollow sphere of radius a. Find the potential $V_0(r, \theta)$ *inside* the hollow sphere. The space inside the sphere is free of charges.

   [*Hint:* Express $V_0(\theta)$ as a function of $\cos\theta$ and use the expression for the *general* solution to Laplace's equation in spherical polar coordinates under symmetry in ϕ, as obtained in Lecture 4.]

5. Find the magnetic dipole moment $\boldsymbol{m}$ of the following objects and thus find the dipole component of the vector potential $\boldsymbol{A}_{\text{dipole}}$.

   [N.B. Remember to specify the directions of $\boldsymbol{m}$ and $\boldsymbol{A}_{\text{dipole}}$ as well as the magnitudes.]

   In both cases the object carries a uniform charge density, the centre of the object lies at the origin, and the object is rotating about the z axis with angular frequency ω.

(a) A spherical shell of radius $\mathcal{K}$ with uniform surface charge density σ.

(b) A sphere of radius $\mathcal{K}$ with uniform volume charge density ρ.

N.B. The "from first principles" way to solve this latter problem is to define an infinitesimal element of volume within the sphere, work out the contribution $\mathrm{d}m$ from this volume element and then integrate over the whole volume of the sphere to find m. However, *in addition,* you might like to try the following alternative approaches to get the same answer:

i. Make use of the result for a spherical shell of charge [from part (a) of this question].

ii. Make use of the result for a flat disk of charge [from Q2 of the Revision Examples Class from Week 3].

6. Verify by explicit differentiation that

$$\boldsymbol{\nabla}_{r'}\left(\frac{1}{R}\right) = -\boldsymbol{\nabla}\left(\frac{1}{R}\right),$$

where $\boldsymbol{\nabla}_{r'}$ represents differentiation with respect to the primed coordinates and $\boldsymbol{\nabla}$ represents (as usual) differentiation with respect to the unprimed coordinates.

7. An infinitely long, solid, conducting cylinder of radius a is coaxial with the $\hat{\boldsymbol{z}}$ axis and rotates with constant angular velocity ω about this axis. A constant, uniform, external magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ is applied. The total charge on the cylinder is zero. You should assume that (i) the vacuum permittivity, ϵ_0 can be used throughout, including inside the cylinder, (ii) any magnetic field due to the rotating cylinder can be neglected, (iii) that the system has reached a steady state.

(a) Working in cylindrical coordinates (s, ϕ, z) find the electric field vector $\boldsymbol{E}$ within the cylinder.

(b) Determine the electrostatic potential $V(r, \phi)$ within the cylinder $(s < a)$.

(c) Determine the volume charge density, ρ, within the cylinder and hence show that it is a constant, independent of s and ϕ.

(d) Determine the surface charge density σ, at the surface of the cylinder $(s = a)$.

(e) Determine the electric field and potential outside the cylinder $(s > a)$.

(f) Carry out any possible cross checks/consistency checks you can devise of the results you have obtained.