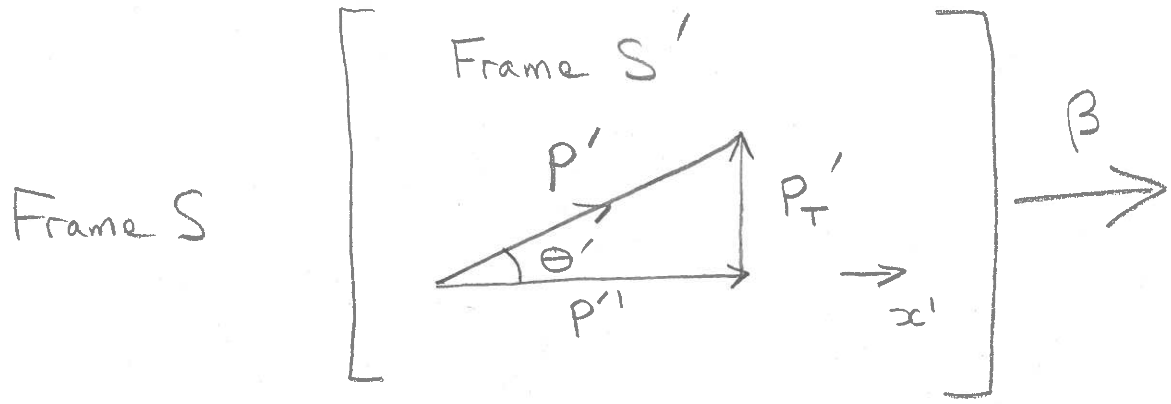
PHYS30441

Electrodynamics

Additional Examples 2

Solutions - Terry Wyatt.

Q1)



photon has zero rest mass $\frac{E'}{c} = p'$

$$p'^{\mu} = [p', \underline{p}']$$

$$p'^{1} = p'\cos\theta'$$

(a) $p^{0} = \gamma(p' + \beta p'^{1}) = \gamma p'(1 + \beta\cos\theta') = p$

$$p^{1} = \gamma(\beta p' + p'^{1}) = \gamma p'(\beta + \cos\theta')$$

$$P_T = P_T' = p'\sin\theta'$$

(b) $\cos\theta = \frac{p^{1}}{p} = \frac{\beta + \cos\theta'}{1 + \beta\cos\theta'}$

(c) In frame S': β'_x, the speed of γ in x' direction $= \cos\theta'$
In frame S: β_x, the speed of γ in x' direction $= \cos\theta$
in units of c

∴ Answer to (b) is of form $\beta_x = \frac{\beta + \beta'_x}{1 + \beta\beta'_x}$,
the usual equation for adding βs!

(d) Cross Check

$$\underset{\sim}{p}^2 = \left(p^0\right)^2 - \left(p^1\right)^2 - \left(p^2\right)^2$$

$$= \left(\gamma p'\right)^2 \left[\left(1 + \beta \cos\theta'\right)^2 - \left(\beta + \cos\theta'\right)^2 \right] - \left(p' \sin\theta'\right)^2$$

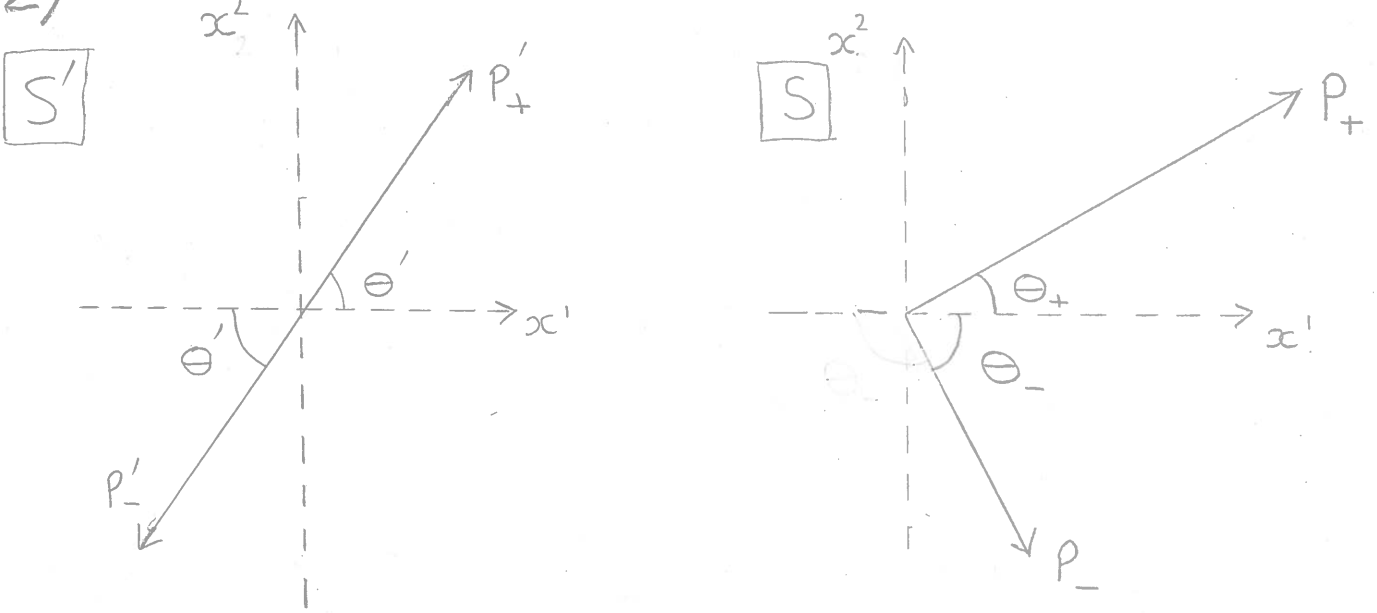
$$= \left(\gamma p'\right)^2 \left[1 + \beta^2 \cos^2\theta' + \cancel{2\beta\cos\theta'} - \left\{ \beta^2 + \cos^2\theta' + \cancel{2\beta\cos\theta'} \right\} - \frac{\sin^2\theta'}{\gamma^2} \right]$$

$$= \left(\gamma p'\right)^2 \left[\left(1 - \beta^2\right) \left\{ \underbrace{1 - \cos^2\theta' - \sin^2\theta'}_{=0} \right\} \right]$$

$$= 0 \quad \checkmark$$

Q2)

(a)



(b) $$\underset{\sim}{p}'_+ = (p', \; p'\cos\theta', \; p'\sin\theta', \; 0)$$

$$\underset{\sim}{p}'_- = (p', \; -p'\cos\theta', \; -p'\sin\theta', \; 0)$$

(c) $$\underset{\sim}{p}_+ = \left(\gamma p'[1+\beta\cos\theta'], \; \gamma p'[\beta+\cos\theta'], \; p'\sin\theta', \; 0\right)$$

$$\underset{\sim}{p}_- = \left(\gamma p'[1-\beta\cos\theta'], \; \gamma p'[\beta-\cos\theta'], \; -p'\sin\theta', \; 0\right)$$

(d) $$\Delta = \theta_+ + \theta_-$$

$$\cos\Delta = \frac{\underline{p}_+ \cdot \underline{p}_-}{p_+ \, p_-} = \frac{\gamma^2(\beta+\cos\theta')(\beta-\cos\theta') - \sin^2\theta'}{\gamma^2(1+\beta\cos\theta')(1-\beta\cos\theta')}$$

$$\therefore \; \Delta = \cos^{-1}\left[\frac{\beta^2 - \cos^2\theta' - \sin^2\theta'/\gamma^2}{1-\beta^2\cos^2\theta'}\right]$$

Q3) $\underline{B} = \nabla \times \underline{A}$

$$\int_S \underline{B} \cdot \underline{da} = \int_S (\nabla \times \underline{A}) \cdot dA$$

$$= \oint \underline{A} \cdot \underline{dl}$$ by Stokes's theorem.

Question 4)

$$V_0 = k\sin^2\frac{\theta}{2} = \frac{k}{2}\left(1-\cos\theta\right)$$

The general solution to Laplace's equation in (r,θ) is:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

On physical grounds $B_l = 0$ $(\forall l)$ to avoid $V\to\infty$ at $r\to 0$, since the region inside the hollow sphere is free of charges.

$$\therefore \quad \frac{k}{2}\left(1-\cos\theta\right) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

We can evaluate the A_l "by inspection" at $r=a$, by equating coefficients of powers of $\cos\theta$.

$l=0$: $P_0 = 1$; $\frac{k}{2} = A_0$

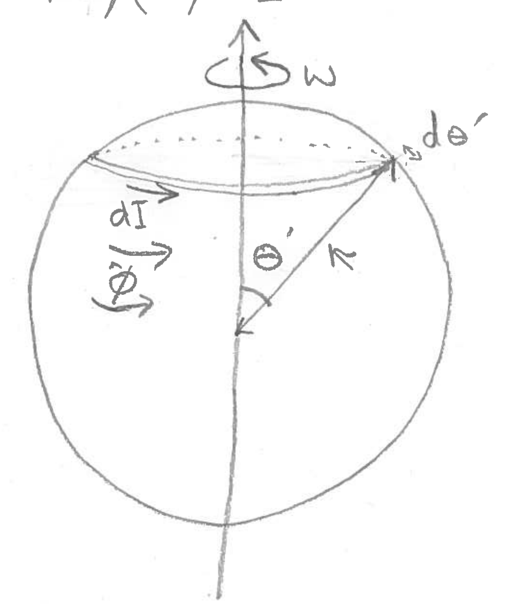
$l=1$: $P_1 = \cos\theta$; $-\frac{k}{2} = A_1 a$

(and $A_l = 0$ otherwise)

Therefore inside the sphere the potential is:

$$V(r,\theta) = \frac{k}{2}\left(1-\frac{r}{a}\cos\theta\right)$$

Q5)(a)



Consider surface element at angle θ, width $K\,d\theta'$, and speed $v = wK\sin\theta$

Current: $dI = \sigma(K\,d\theta')\,wK\sin\theta'$
$= \sigma w K^2 \sin\theta'\,d\theta'$

Area enclosed: $a = \pi(K\sin\theta')^2$

Magnetic dipole moment:

$$m = \int a\,dI = \pi\sigma w K^4 \int_0^\pi \sin^3\theta'\,d\theta'$$

$$= \pi\sigma w K^4 \int_0^\pi (\cos^2\theta' - 1)\,d(\cos\theta')$$

$$= \pi\sigma w K^4 \left[\frac{\cos^3\theta'}{3} - \cos\theta\right]_0^\pi = \pi\sigma w K^4\left[\left[-\tfrac{1}{3}+1\right] - \left[\tfrac{1}{3}-1\right]\right]$$

$$\therefore\ \underline{m} = \frac{4}{3}\pi\sigma w K^4\,\hat{\underline{z}}$$

$$\underline{A}_{\text{dipole}} = \frac{\mu_0}{4\pi}\frac{\underline{m}\times\hat{\underline{r}}}{r^2} = \frac{\mu_0\sigma w K^4}{3r^2}\sin\theta\,\hat{\underline{\phi}}$$

Q5(b)

Consider infinitessimal volume element $d\tau$ VP

Q8)(b)

$$d\tau = r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$dq = \rho \, d\tau$$

$$dI = \frac{\omega}{2\pi} dq$$

$$dm = \pi (r\sin\theta)^2 dI = \frac{\rho\omega}{2} r^4 \sin^3\theta \, d\phi \, d\theta \, dr$$

$$m = \frac{\rho\omega}{2} \int_0^{2\pi} d\phi \int_0^R r^4 dr \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{\rho\omega}{2} \, 2\pi \, \frac{R^5}{5} \underbrace{\int_{-1}^{1} (1-\cos^2\theta) \, d(\cos\theta)}$$

$$\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_{-1}^{1} = \frac{2}{3} - \left(\frac{-2}{3}\right)$$

$$\underline{m} = \frac{4\pi}{15} \rho\omega R^5 \underline{\hat{z}}$$ (in the $\hat{z}$ direction, e.g., from the right hand rule)

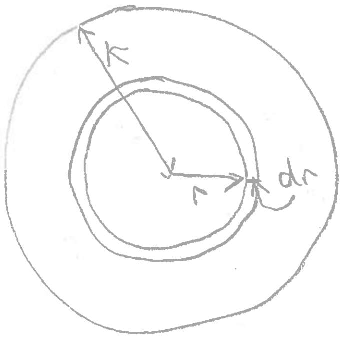
$$\underline{A}_{dipole} = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{\underline{r}}}{r^2}$$

$$= \frac{\mu_0}{4\pi} \frac{|m|}{r^2} \sin\theta \, \hat{\underline{\phi}}$$

$$= \frac{\mu_0}{15} \frac{\rho \omega R^5}{r^2} \sin\theta \, \hat{\underline{\phi}}$$

Q5) (b) [continued] Here are a couple of alternative ways to solve this problem that make use of results we've obtained previously.

(i) Divide sphere into an infinite set of spherical shells, as considered in part (a) of this question.



surface charge:

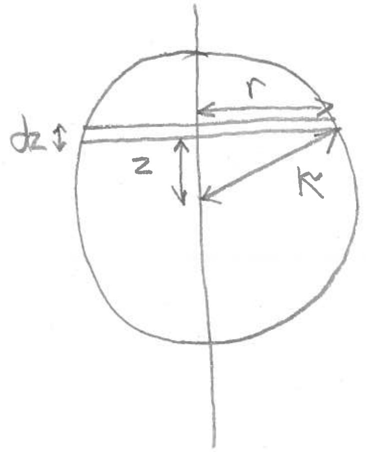
$$\sigma \Rightarrow \rho\, dr$$

$$\frac{4}{3}\pi\sigma\omega K^4 \Rightarrow \frac{4}{3}\pi\rho\omega r^4\, dr$$

For the sphere:

$$m = \frac{4}{3}\pi\rho\omega \int_0^K r^4\, dr = \frac{4}{15}\pi\rho\omega K^5$$

(ii) Divide sphere into an infinite set of flat disks, as considered in the Revision Examples Class Q2:



surface charge:

$$\sigma \Rightarrow \rho \, dz$$

$$\frac{\pi}{4}\sigma\omega R^4 \Rightarrow \frac{\pi}{4}\rho\omega r^4 \, dz$$

$$r^2 = R^2 - z^2$$

For the sphere:

$$m = \frac{\pi}{4}\rho\omega \int_{-R}^{R} \left(R^4 - 2R^2z^2 + z^4\right) dz$$

$$= \frac{\pi}{4}\rho\omega \left[R^4 z - \frac{2}{3}R^2 z^3 + \frac{z^5}{5}\right]_{-R}^{R}$$

$$= \frac{\pi}{4}\rho\omega . 2R^5 \underbrace{\left[1 - \frac{2}{3} + \frac{1}{5}\right]}_{8/15} = \frac{4}{15}\pi\rho\omega R^5$$

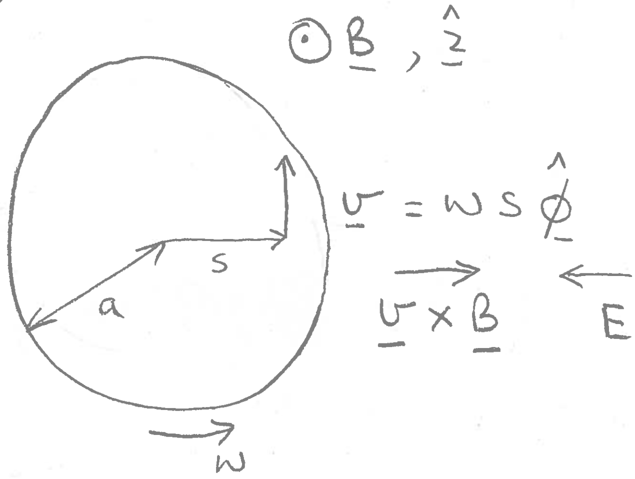
Q6)

$$\nabla_{r'}\left(\frac{1}{R}\right) = \left(\hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}\right)\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}$$

$$= -\frac{1}{2}\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\left[-\hat{x}\,2(x-x')-\hat{y}\,2(y-y')-\hat{z}\,2(z-z')\right]$$

$$= \frac{\mathbf{R}}{R^3} = \underbrace{\frac{\hat{R}}{R^2} = -\nabla\left(\frac{1}{R}\right)}_{\text{as we showed in Lecture 1}}$$

Q7)



(a) In steady state $0 = F = q(\underline{E} + \underline{v} \times \underline{B})$

$$\therefore \underline{E} = -\underline{v} \times \underline{B} = -\omega B s \,\hat{\underline{s}} \quad \text{inside cylinder}$$

(b) Since $\underline{E} = -\nabla V$ and in cylindrical coordinates

$$\nabla V = \frac{\partial V}{\partial s}\hat{\underline{s}} + \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\underline{\phi}} + \frac{\partial V}{\partial z}\hat{\underline{z}}$$

it can be easily seen by inspection that

$$V = \frac{\omega B s^2}{2} + V_0 \quad , \quad \text{where } V_0 \text{ is a constant.}$$

inside cylinder

Q7)

(c) $\rho = \varepsilon_0 \nabla \cdot \underline{E} = \varepsilon_0 \frac{1}{s}\frac{\partial}{\partial s}\left(s E_s\right)$ since $\underline{E}$ has only $\hat{s}$ component

$= \varepsilon_0 \frac{1}{s}\frac{\partial}{\partial s}\left(-\omega B s^2\right) = -2\varepsilon_0 \omega B$ inside the cylinder

That is ρ is independent of s and ϕ.

(d) In order to ensure that the cylinder is uncharged there must be a charge at the surface with density σ such that

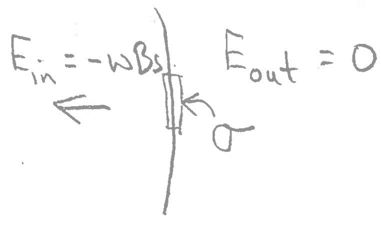
$\pi a^2 \rho = -2\pi a \sigma$ charge per unit length of cylinder

$\therefore \sigma = -\frac{a\rho}{2} = a\varepsilon_0 \omega B$

(e) Given the cylindrical symmetry of the charge distributions Gauss's Law tells us that outside the cylinder: $E = 0$ and V is constant.

(f) Cross check results by considering discontinuity of $\underline{E}$ at the surface of the cylinder

$\frac{\sigma}{\varepsilon_0} = E_{out} - E_{in}$



$\frac{a\varepsilon_0 \omega B}{\varepsilon_0} = 0 - (-\omega B a)$ ✓