Electrodynamics (PHYS30441) Additional Examples 3

1. The statement of local charge conservation (or the continuity equation for electric charge) may be written in Lorentz covariant form as:
$$\frac{\partial j^\mu}{\partial x^\mu} = \partial_\mu j^\mu = 0,$$
where $j^\mu = (c\rho, \boldsymbol{j})$ is the (contravariant) 4-vector current density. As we discussed in the lectures, the fact that we can write this relation as the product of a covariant (∂_μ) and a contravariant (j^μ) 4-vector guarantees that the equation is Lorentz covariant (Lorentz invariant in form). But let's verify explicitly that this is, indeed, the case by using the substitutions $\partial'_\mu = \left(\Lambda^{-1}\right)^\nu{}_\mu\, \partial_\nu$ and $j'^\mu = \Lambda^\mu{}_\nu j^\nu$. Verify that $\partial'_\mu j'^\mu = 0$ in two ways:
   (a) using index notation for matrix multiplication *[N.B. be careful that any expression you write down has no more than two occurrences of the same Lorentz index.]*
   (b) (more long-winded, but instructive to do once in your life ;-) by writing out explicitly the components of ∂'_μ and j'^μ and multiplying.

2. The 4×4 matrix $\Lambda^\mu{}_\nu(\beta)$ represents a Lorentz boost along the x^1-axis. The results of two successive Lorentz boots by β_1 and β_2 along the x^1-axis can be represented by the product of the relevant matrices. Verify that
$$\Lambda^\mu{}_\kappa(\beta_1)\Lambda^\kappa{}_\nu(\beta_2) = \Lambda^\mu{}_\nu(\beta_3),$$
where β_3 is given by the standard relativistic result for adding two velocities in same direction (e.g. as you obtained in answer to Question 1(b) in examples class 2).

3. Show that the Lorentz transformation matrix
$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
may be represented equivalently by the matrix
$$M^\mu{}_\nu = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
where $\tanh\eta = \beta$. This corresponds to a *hyperbolic rotation* through an angle η within the two-dimensional Minkowski sub-space (x^0, x^1).

   The parameter η is called the *rapidity* in particle physics. Making use of your answers to question 2 and by considering a formula for $\tanh(\eta_1 + \eta_2)$, show that the rapidities for two successive Lorentz boosts along the same axis add linearly.

4. In Lecture 11 we wrote down Lorentz transformations using the matrices $\Lambda^\mu{}_\nu$ and $\left(\Lambda^{-1}\right)^\mu{}_\nu$. Note that in both cases the left-hand index (which labels rows of the matrix) is an upper (or contravariant) index and the right-hand index (which labels columns of the matrix) is a lower (or covariant) index.

   Please note that it is important to maintain this pattern. $\Lambda^\mu{}_\nu$ is a symmetric matrix and therefore $\Lambda^\mu{}_\nu = \Lambda^\nu{}_\mu$. Nevertheless $\Lambda^\mu{}_\nu \neq \Lambda_\nu{}^\mu$.

   By writing $\Lambda_\nu{}^\mu = g_{\nu\alpha}\Lambda^\alpha{}_\beta g^{\beta\mu}$ and multiplying out the matrices show that $\Lambda_\nu{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\nu$!