PHYS30441

Electrodynamics

Additional Examples 3

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Q1)(a) As we showed in the lectures, differentials with respect to a contravariant 4-vector transform as a covariant:

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu \partial_\nu = \partial_\nu (\Lambda^{-1})^\nu{}_\mu$$

$$\therefore \quad \partial'_\mu j'^\mu = (\Lambda^{-1})^\lambda{}_\mu \partial_\lambda \Lambda^\mu{}_\nu j^\nu$$

$$= \partial_\lambda (\Lambda^{-1})^\lambda{}_\mu \Lambda^\mu{}_\nu j^\nu$$

$$= \partial_\lambda \delta^\lambda{}_\nu j^\nu$$

$$= \partial_\nu j^\nu = 0$$

That is, the continuity equation for charge is valid in all frames related by a Lorentz Transformation: it is Lorentz covariant.

Q1)(b) Complementary approach, writing out transformations explicitly.

$$\partial'_\mu = (\Lambda^{-1})^\nu{}_\mu \partial_\nu$$

$$= (\gamma\partial_0 + \gamma\beta\partial_1,\ \gamma\beta\partial_0 + \gamma\partial_1,\ \partial_2,\ \partial_3)$$

$$j'^\mu = \Lambda^\mu{}_\nu j^\nu$$

$$= (\gamma j^0 - \gamma\beta j^1,\ -\gamma\beta j^0 + \gamma j^1,\ j^2,\ j^3)$$

$$\therefore\ \partial'_\mu j'^\mu = \gamma^2(\partial_0 j^0 - \beta^2 \partial_1 j^1 - \beta\partial_0 j^1 + \beta\partial_1 j^0$$
$$- \beta^2\partial_0 j^0 + \partial_1 j^1 + \beta\partial_0 j^1 - \beta\partial_1 j^0)$$
$$+ \partial_2 j^2 + \partial_3 j^3$$

$$= \underbrace{\gamma^2(1-\beta^2)}_{=1}(\partial_0 j^0 + \partial_1 j^1) + \partial_2 j^2 + \partial_3 j^3$$

$$= \partial_\mu j^\mu \quad \text{as required.}$$

Q2) $\Lambda^{\mu}{}_{k}(\beta_1)\Lambda^{k}{}_{\nu}(\beta_2)$

$$= \left(\begin{array}{cc|c} \gamma_1 & -\gamma_1\beta_1 & \\ -\gamma_1\beta_1 & \gamma_1 & 0 \\ \hline & 0 & 1_{2\times 2} \end{array}\right)\left(\begin{array}{cc|c} \gamma_2 & -\gamma_2\beta_2 & \\ -\gamma_2\beta_2 & \gamma_2 & 0 \\ \hline & 0 & 1_{2\times 2} \end{array}\right)$$

$$= \left(\begin{array}{cc|c} \gamma_1\gamma_2(1+\beta_1\beta_2) & -\gamma_1\gamma_2(\beta_1+\beta_2) & \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 \\ \hline & 0 & 1_{2\times 2} \end{array}\right)$$

If this matrix is to be equal to $\Lambda^{\mu}{}_{\nu}(\beta_3)$, as claimed in the question, we require

$$\gamma_3 = \gamma_1\gamma_2(1+\beta_1\beta_2)$$
$$-\gamma_3\beta_3 = -\gamma_1\gamma_2(\beta_1+\beta_2)$$

Dividing these expressions gives

$$\beta_3 = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$, as obtained previously in (a).

Q2 (continued)

In addition, we need to show that

$$\gamma_3 = \gamma_1 \gamma_2 (1 + \beta_1 \beta_2)$$

Since $\gamma_3 \equiv (1 - \beta_3^2)^{-1/2}$

$$= \left[1 - \left(\frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}\right)^2\right]^{-1/2}$$

$$= (1 + \beta_1 \beta_2)\left[(1 \cancel{+2\beta_1\beta_2} + \beta_1^2 \beta_2^2) - (\beta_1^2 \cancel{+2\beta_1\beta_2} + \beta_2^2)\right]^{-1/2}$$

$$= (1 + \beta_1 \beta_2) \underbrace{(1 - \beta_1^2)^{-1/2}}_{\gamma_1} \underbrace{(1 - \beta_2^2)^{-1/2}}_{\gamma_2}$$

as required

Q3) Using the standard result:

$$\cosh^2\eta - \sinh^2\eta = 1$$

$$1 - \tanh^2\eta = \frac{1}{\cosh^2\eta}$$

$$\therefore \quad \cosh\eta = \frac{1}{(1-\tanh^2\eta)^{1/2}} = \frac{1}{(1-\beta^2)^{1/2}} = \gamma,$$

as required for the $(0,0)$ elements of $\Lambda^\mu{}_\nu$, $M^\mu{}_\nu$ to be equal

$$-\sinh\eta = -\tanh\eta\cosh\eta = -\beta\gamma$$

as required for the $(0,1)$ elements to be equal.

Q3) (continued)

Starting from (or deriving) the standard formula

$$\tanh(\eta_1 + \eta_2) = \frac{\tanh\eta_1 + \tanh\eta_2}{1 + \tanh\eta_1 \tanh\eta_2}$$

and substituting for

$$\tanh\eta_1 = \beta_1$$

$$\tanh\eta_2 = \beta_2$$

gives

$$\tanh(\eta_1 + \eta_2) = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \beta_3 \quad \left\{\begin{matrix}\text{from answer} \\ \text{to Q1}\end{matrix}\right\}$$

$\therefore\ \eta_3 = \eta_1 + \eta_2$ as required

Q4)

$$\Lambda_\mu{}^\nu = g_{\mu\alpha}\Lambda^\alpha{}_\beta\, g^{\beta\nu}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & & 0 \\ -\gamma\beta & -\gamma & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & \gamma\beta & & 0 \\ \gamma\beta & \gamma & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} = (\Lambda^{-1})^\mu{}_\nu$$