Electrodynamics (PHYS30441) Additional Examples 4

1. In Lectures 14 and 15 we obtained the 4-potential A^μ and the $\boldsymbol{E}, \boldsymbol{B}$ fields produced by a moving point charge by (a) writing down the components of the 4-potential and fields in the rest frame S' of the charge and (b) applying the appropriate Lorentz transformations into a frame (S) in which the charge is moving with constant velocity along the x^1 axis.

   In this problem we shall use an analogous approach to find the 4-potential and $\boldsymbol{E}, \boldsymbol{B}$ fields produced by an infinitely long beam of positrons with radius a that lies along the x^1 axis and moves with constant velocity along the x^1 axis.

   Let's first consider the beam of positrons in its rest frame, S'. In the sheet from week 3 "Additional Revision Problems", Question 10 we found expressions for the electric field [in part (a)] and the scalar potential [in part (c)] produced by the beam of positrons, as a function of s, the radial distance to the axis of the beam. We found separate expressions for positions inside ($s < a$) and outside ($s > a$) the beam. These expressions are valid in the rest frame of the beam, except that we should write the charge density as ρ_0. [*If you did not attempt "Additional Revision Problems", Question 10 in week 3 then I suggest you do at least parts (a) and (c) now, before attempting the current problem.*]

   (a) Use the answers from "Additional Revision Problems", Question 10 (c) to write down the components of the 4-potential A'^μ in frame S', both inside ($s < a$) and outside ($s > a$) the beam.

   [*Note: the expression for the scalar potential outside the beam includes some constant terms. Since these terms are independent of s and thus do not affect the fields, let's just ignore them.*]

   (b) By using an appropriate Lorentz Transformation, find expressions for the 4-potential A^μ in the frame S, in which the beam is moving constant velocity β along the x^1 axis. Do this for both inside ($s < a$) and outside ($s > a$) the beam. Give a physical interpretation for the factor of γ that appears in the relationship between the scalar potentials in the two frames of reference.

   (c) Hence, use the definition of the electromagnetic field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ to find the $\boldsymbol{E}, \boldsymbol{B}$ fields produced by the moving beam. Do this for both inside ($s < a$) and outside ($s > a$) the beam.

   (d) Use the answers from "Additional Revision Problems", Question 10 (a) to show that the same results for $\boldsymbol{E}, \boldsymbol{B}$ in S can be obtained by applying the appropriate transformation equations to the fields $\boldsymbol{E'}$ and $\boldsymbol{B'}$ in S'.

   (e) Apply the Lorentz-covariant equation that describes the interaction of a charged particle with the fields:

   $$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = q\,F^{\mu\nu}\,u_\nu,$$

   to this system for the region inside the beam ($s < a$). Do this both for frame S' and for frame S. Give a physical interpretation for the results you obtain. Comment on the behaviour as $\beta \to 1$ and compare this with the behaviour discussed in the solution to "Additional Revision Problems", Question 10 (f).

Answers:

	Inside		Outside	
a)	$A'^0 = \dfrac{-s^2\rho_0}{4\epsilon_0 c}$		$A'^0 = \dfrac{-a^2\rho_0 \ln s}{2\epsilon_0 c}$	
b)	$A^0 = \dfrac{-\gamma s^2\rho_0}{4\epsilon_0 c}$,	$A^1 = \dfrac{-\beta\gamma s^2\rho_0}{4\epsilon_0 c}$	$A^0 = \dfrac{-\gamma a^2\rho_0 \ln s}{2\epsilon_0}$,	$A^1 = \dfrac{-\beta\gamma a^2\rho_0 \ln s}{2\epsilon_0 c}$
c)	$\boldsymbol{E} = \dfrac{\gamma\rho_0}{2\epsilon_0}\boldsymbol{s}$		$\boldsymbol{E} = \dfrac{\gamma\rho_0 a^2}{2\epsilon_0 s}\hat{\boldsymbol{s}}$	
	$\boldsymbol{B} = \dfrac{\beta\gamma\rho_0 s}{2\epsilon_0 c}\hat{\boldsymbol{\phi}}$		$\boldsymbol{B} = \dfrac{\beta\gamma\rho_0 a^2}{2\epsilon_0 c s}\hat{\boldsymbol{\phi}}$	
e)	$\dfrac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \dfrac{q\rho_0}{2\epsilon_0}\left[0, 0, x^2, x^3\right]$			

2. *"Optional" extra practice for using index notation.*

A useful alternative way of expressing the E and B fields is the so-called "dual field tensor", $\mathcal{F}^{\mu\nu}$, which is defined by

$$\mathcal{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}.$$

Here $\epsilon^{\mu\nu\alpha\beta}$ is the totally anti-symmetric tensor of rank four. It is defined by $\epsilon^{0123} = 1$ and for any even number of permutations, such as, $\epsilon^{1032} = \epsilon^{2130} = \epsilon^{2013} = 1$. $\epsilon^{\mu\nu\alpha\beta}$ changes sign for an odd number of permutations of the indices, such as, $\epsilon^{1023} = \epsilon^{1230} = \epsilon^{2031} = -1$. Of course, this implies that $\epsilon^{\mu\nu\alpha\beta} = 0$ if any pair of the four indices are equal.

In examples class 3, question 2 (b) (ii), you showed that the elements of $F_{\mu\nu} = g_{\mu\alpha}F^{\alpha\beta}g_{\beta\nu}$ are the same as those of $F^{\mu\nu}$ except for the replacement $E_i \to -E_i$. (The signs of the B_i terms are unchanged.) That is,

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{bmatrix}.$$

(a) Using the above definitions, show that the dual field tensor may be written in matrix form as

$$\mathcal{F}^{\mu\nu} = \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3/c & -E_2/c \\ B_2 & -E_3/c & 0 & E_1/c \\ B_3 & E_2/c & -E_1/c & 0 \end{bmatrix}.$$

That is, $\mathcal{F}^{\mu\nu}$ is obtained from $F^{\mu\nu}$ by making the replacements $E_i/c \longrightarrow B_i$ and $B_i \longrightarrow -E_i/c$.

Note: This is a great example of how using index notation can make what looks like an absolutely horrendous problem (multiplying a 2-dimensional matrix by a 4-dimensional matrix) into a relatively simple exercise! Index notation allows us to exploit the fact that $F^{\mu\nu}$ (and $\mathcal{F}^{\mu\nu}$) have only six independent elements and that $\epsilon^{\mu\nu\alpha\beta}$ is a very sparse matrix! Just write down an explicit expression for each of the six independent elements of $\mathcal{F}^{\mu\nu}$ using matrix notation. You'll find that there are very few non-zero terms in the sums over the two free indices!

(b) Verify that the two homogeneous field (Maxwell) equations (that is the two that do not involve the sources) may be written using the dual field tensor in the nice compact form

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0.$$

This problem is "optional" in the sense that I won't ask you any questions that require the "dual field tensor" in the exam. However I suggest making use of the opportunity for extra practice using index notation.

Optional reading: Jackson "Classical Electrodynamics" Section 6.11 has a discussion of duality transformations in the context of electrodynamics.