PHYS30441

Electrodynamics

Additional Examples 4

Solutions – Terry Wyatt.

Q1 (a) <u>Inside</u> | <u>Outside</u>

<u>In frame S'</u>

Inside:

$$A'^0 = \frac{V}{c} = -\frac{s^2\rho_0}{4\varepsilon_0 c}$$

$$A'^1 = A'^2 = A'^3 = 0$$

Outside:

$$A'^0 = -\frac{a^2\rho_0 \ln s}{2\varepsilon_0 c}$$

(b) <u>In frame S</u>

$$A^0 = \gamma\left(A'^0 + \beta A'^1\right) = \gamma A'^0$$

$$A^1 = \gamma\left(\beta A'^0 + A'^1\right) = \gamma\beta A'^0$$

$$A^2 = A^3 = 0 \qquad (s)^2 = (x^2)^2 + (x^3)^2$$

Inside:

$$\frac{\partial (s)^2}{\partial x^i} = 2x^i \text{ for } i = 2 \text{ or } 3$$

$$= 0 \text{ for } i = 0 \text{ or } 1$$

(needed later)

Outside:

$$\frac{\partial[\ln s]}{\partial x^i} = \frac{1}{2}\frac{1}{s^2}2x^i = \frac{x^i}{s^2}$$

for $i = 2$ or 3

and $= 0$ for $i = 0$ or 1.

Inside:

$$A^0 = -\frac{\gamma\rho_0 s^2}{4\varepsilon_0 c}$$

$$A^1 = -\frac{\beta\gamma\rho_0 s^2}{4\varepsilon_0 c}$$

Outside:

$$A^0 = \frac{-a^2\gamma\rho_0 \ln s}{2\varepsilon_0 c}$$

$$A^1 = -\frac{a^2\beta\gamma\rho_0 \ln s}{2\varepsilon_0 c}$$

The expressions for A^0 is the same as for A'^0, except that ρ_0 becomes $\gamma\rho_0$ which is the charge density in the frame S in which beam is moving.

Q1 (c)

$$\frac{E_i}{c} = F^{i0} = \partial^i A^0 - \underbrace{\partial^0 A^i}_{0}$$

For $i = 2, 3$

Inside

$$\frac{E_i}{c} = \frac{-\gamma\rho_0}{4\varepsilon_0 c}\left(\frac{-\partial (s)^2}{\partial x^i}\right)$$

$$= \frac{\gamma\rho_0 x^i}{2\varepsilon_0 c}$$

or $\underline{E} = \dfrac{\gamma\rho_0 \underline{s}}{2\varepsilon_0}$

Outside

$$\frac{E_i}{c} = \frac{-a^2\gamma\rho_0}{2\varepsilon_0 c}\left(\frac{-\partial[\ln s]}{\partial x^i}\right)$$

$$= \frac{a^2\gamma\rho_0}{2\varepsilon_0 c}\frac{x^i}{s^2}$$

or $\underline{E} = \dfrac{a^2\gamma\rho_0 \hat{\underline{s}}}{2\varepsilon_0 s}$

[As found in "Additional Revision Probs" Q 10(a) with $\rho = \gamma\rho_0$]

$E_1 = 0$ everywhere since $\dfrac{\partial s}{\partial x^1} = 0$

Q1 (c) (contd)

Inside | **Outside**

$$B_2 = F^{13} = \underbrace{\partial^1 A^3}_{=0} - \partial^3 A^1$$

Inside:

$$B_2 = \frac{-\beta\gamma\rho_0}{4\varepsilon_0 c}\left[-\left(-\frac{\partial(s)^2}{\partial x^3}\right)\right]$$

$$= -\frac{\beta\gamma\rho_0 x^3}{2\varepsilon_0 c}$$

Outside:

$$B_2 = -\frac{a^2\beta\gamma\rho_0}{2\varepsilon_0 c}\left[-\left(-\frac{\partial[\ln s]}{\partial x^3}\right)\right]$$

$$= -\frac{a^2\beta\gamma\rho_0}{2\varepsilon_0 c}\frac{x^3}{s^2}$$

$$B_3 = F^{21} = \partial^2 A^1 - \underbrace{\partial^1 A^2}_{=0}$$

Inside:

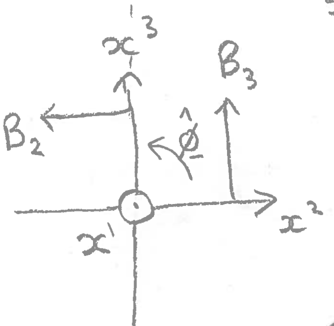
$$B_3 = -\frac{\beta\gamma\rho_0}{4\varepsilon_0 c}\left(-\frac{\partial(s)^2}{\partial x^2}\right)$$

$$= \frac{\beta\gamma\rho_0}{2\varepsilon_0 c}x^2$$

Outside:

$$B_3 = -\frac{a^2\beta\gamma\rho_0}{2\varepsilon_0 c}\left(-\frac{\partial[\ln s]}{\partial x^2}\right)$$

$$= \frac{a^2\beta\gamma\rho_0}{2\varepsilon_0 c}\frac{x^2}{s^2}$$



or $\underline{B} = \dfrac{\beta\gamma\rho_0 s}{2\varepsilon_0 c}\hat{\underline{\phi}}$ | or $\underline{B} = \dfrac{a^2\beta\gamma\rho_0}{2\varepsilon_0 c s}\hat{\underline{\phi}}$

These are constent with "Additional Revision Probs" Q10(b) if we note that $\gamma\rho_0 = \rho$ and $\dfrac{\beta}{\varepsilon_0 c} = \mu_0 c\beta = \mu_0 v$. In addition, $\hat{\underline{\phi}}$ has here been defined as rotation about direction of $\underline{\beta}$

Q1 (d)

In frame S' the only non-zero components of $F'^{\mu\nu}$ are E_2' and E_3'

Using the transformation equations (Lecture 13) we can write the non-zero components of $F^{\mu\nu}$

$$\frac{E_2}{c} = \frac{\gamma E_2'}{c} \quad , \quad \frac{E_3}{c} = \frac{\gamma E_3'}{c}$$

$$B_2 = -\frac{\gamma\beta E_3'}{c} \quad , \quad B_3 = \frac{\gamma\beta E_2'}{c}$$

which are consistent with the results in (c)

Note that we can write also that

$$\underline{B} = \frac{1}{c}\left(\underline{\beta} \times \underline{E}\right)$$

in frame S (and also in frame S')

Q1 (e) (Inside the beam)

Note minus sign because covariant u_ν

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu \,, \quad \text{where } u_\nu = \gamma c\,[1, -\underline{\beta}]$$

$$= \frac{q\gamma\rho_0}{2\varepsilon_0 c}\begin{bmatrix} 0 & 0 & -x^2 & -x^3 \\ 0 & 0 & -\beta x^2 & -\beta x^3 \\ x^2 & \beta x^2 & 0 & 0 \\ x^3 & \beta x^3 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -\beta \\ 0 \\ 0 \end{bmatrix}\gamma c$$

$$= \frac{q\gamma^2\rho_0}{2\varepsilon_0}\begin{bmatrix} 0 \\ 0 \\ x^2-\beta^2x^2 \\ x^3-\beta^2x^3 \end{bmatrix} = \frac{q\rho_0}{2\varepsilon_0}\begin{bmatrix} 0 \\ 0 \\ x^2 \\ x^3 \end{bmatrix}$$

Does this tie in with our earlier results?

In the rest frame S' we have $\Delta t' = \Delta\tau$ and we recover the expected non-relativistic result

$$\underline{F}' = \frac{d\underline{p}'}{dt'} = \frac{d\underline{p}'}{d\tau} = q\frac{\rho_0}{2\varepsilon_0}\underline{s}' = q\underline{E}'$$

$\left\{\text{where } \underline{s}' = [0, x'^2, x'^3]\right\}$

$$\frac{d\mathcal{E}'}{dt'} = q\underline{v}'\cdot\underline{E}' = 0$$

Q1 (e) (contd.)

In frame S: $p^2 = p'^2$, $p^3 = p'^3$,

$x^2 = x'^2$, $x^3 = x'^3$,

since transverse components of 4-vectors are unaffected by L.T.

$\Delta t = \gamma \Delta \tau$

Looking at the transverse $\mu = 2, 3$ components we have:

$$\therefore \frac{dp_\perp^\mu}{dt} = \frac{1}{\gamma}\frac{dp'^\mu}{dt'} = \frac{q\rho_0}{2\varepsilon_0 \gamma}\underline{s},$$

which we can think of as a result of time dilation.

Discussion

In "Additional Revision Problems," Q 10) (f) we obtained the result that inside the beam

$$\frac{dp_\perp}{dt} = F \sim \frac{q\rho}{2}\left(\frac{1}{\varepsilon_0} - v^2\mu_0\right)\underline{s} = \frac{q\gamma\rho_0}{2\varepsilon_0}\left(1 - \frac{v^2}{c^2}\right)\underline{s} = \frac{q\rho_0\,\underline{s}}{2\varepsilon_0\gamma},$$

which, remarkably, is consistent with the correct relativistic result. However, in Q10) (f) we didn't take into account the increase of the charge density, $\rho = \gamma\rho_0$, or time dilation, $\Delta t = \gamma \Delta \tau$ in the frame in which the positrons are moving. The fact that the two associated factors of γ cancel is perhaps just a coincidence?

Certainly if we had expressed our answer in Q10) (f) in terms of acceleration we would have got the wrong result from the non-relativistic treatment!

The correct relativistic treatment gives in frame S

$$\frac{dp_\perp}{dt} = \frac{d}{dt}\left(\gamma m \underline{v}\right) = \underset{\uparrow}{\gamma} m \frac{d\underline{v}}{dt} = \gamma m \underline{a} = \frac{q \rho_0}{2\varepsilon_0 \gamma} \underline{s}$$

N.B. Only possible when acceleration, a, is transverse to velocity!

$$\therefore \quad \underline{a} = \frac{q\rho_0}{2\varepsilon_0 \gamma^2 m} \underline{s} = \frac{\underline{a}'}{\gamma^2},$$

as we expect from our previous general treatment of acceleration!

Q2)

(a) Using $\mathcal{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ and the definitions of $\epsilon^{\mu\nu\alpha\beta}$ and $F_{\alpha\beta}$ given in the problem we have

$$\mathcal{F}^{01} = \frac{1}{2}\left[\epsilon^{01\underset{\sim}{23}} F_{23} + \epsilon^{01\underset{\sim}{32}} F_{32}\right]$$

[N.B. these are the only two possible values of the free indices, because of the anti-symmetric nature of $\epsilon^{\mu\nu\alpha\beta}$!]

$$= \frac{1}{2}\left[(1)(-B_1) + (-1)B_1\right] = -B_1$$

$$\mathcal{F}^{02} = \frac{1}{2}\left[\epsilon^{0213} F_{13} + \epsilon^{0231} F_{31}\right]$$

$$= \frac{1}{2}\left[(-1)B_2 + (1)(-B_2)\right] = -B_2$$

$$\mathcal{F}^{03} = \frac{1}{2}\left[\epsilon^{0312} F_{12} + \epsilon^{0321} F_{21}\right]$$

$$= \frac{1}{2}\left[(1)(-B_3) + (-1)B_3\right] = -B_3$$

Also $\mathcal{F}^{0i} = \epsilon^{0i\alpha\beta} F_{\alpha\beta} = -\epsilon^{i0\alpha\beta} F_{\alpha\beta} = -\mathcal{F}^{i0}$.

Similarly,

$$\mathbb{F}^{12} = \frac{1}{2}\left[\epsilon^{1230} F_{30} + \epsilon^{1203} F_{03}\right]$$

$$= \frac{1}{2}\left[(-1)\left(-\frac{E_3}{c}\right) + (1)\frac{E_3}{c}\right] = \frac{E_3}{c}$$

$$\mathbb{F}^{13} = \frac{1}{2}\left[\epsilon^{1320} F_{20} + \epsilon^{1302} F_{02}\right]$$

$$= \frac{1}{2}\left[(1)\left(-\frac{E_2}{c}\right) + (-1)\frac{E_2}{c}\right] = -\frac{E_2}{c}$$

$$\mathbb{F}^{23} = \frac{1}{2}\left[\epsilon^{2301} F_{01} + \epsilon^{2310} F_{10}\right]$$

$$= \frac{1}{2}\left[(1)\left(\frac{E_1}{c}\right) + (-1)\left(-\frac{E_1}{c}\right)\right] = \frac{E_1}{c}$$

and $\mathbb{F}^{ij} = -\mathbb{F}^{ji}$.

Q2) (b) Consider $\nu = 0$

$$0 = \partial_\alpha F^{\alpha 0} = \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30}$$

$$= 0 + \frac{\partial B_1}{\partial x^1} + \frac{\partial B_2}{\partial x^2} + \frac{\partial B_3}{\partial x^3}$$

$$\therefore \quad \boxed{0 = \nabla . \underline{B}}$$

Consider $\nu = 1$

$$0 = \partial_\alpha F^{\alpha 1} = \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}(-B_1) + 0 + \frac{\partial}{\partial x^2}\left(-\frac{E_3}{c}\right) + \frac{\partial}{\partial x^3}\left(\frac{E_2}{c}\right)$$

$$= -\frac{1}{c}\left[\frac{\partial \underline{B}}{\partial t} + \nabla \times \underline{E}\right]_1$$

$\partial_\alpha F^{\alpha 2}$ and $\partial_\alpha F^{\alpha 3}$ yield similar results.

$$\boxed{0 = \frac{\partial \underline{B}}{\partial t} + \nabla \times E}$$

as required.