PHYS30441

Electrodynamics

Additional Examples 5

Solutions - Terry Wyatt.

Q1) (a) A possible guess might be that Gauss's Law still applies ?

$$\oint \underline{E} \cdot d\underline{a} = \frac{q}{\varepsilon_0} \quad ?$$

However, it's also possible to imagine the odd factor of γ thrown in !?

Q1 (b)

Because of the solid angle term $\sin\theta \, d\theta$
the substitution $u = \cos\theta$, $du = -\sin\theta \, d\theta$
often works in such cases.

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{q(1-v^2/c^2)}{4\pi\epsilon_0} \int \frac{R^2 \sin\theta \, d\theta \, d\phi}{R^2(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}} = \frac{q(1-v^2/c^2)}{4\pi\epsilon_0} 2\pi \int_0^\pi \frac{\sin\theta \, d\theta}{(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}}.$$

Let $u \equiv \cos\theta$, so $du = -\sin\theta d\theta$, $\sin^2\theta = 1-u^2$.

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{q(1-v^2/c^2)}{2\epsilon_0} \int_{-1}^{1} \frac{du}{[1-\frac{v^2}{c^2}+\frac{v^2}{c^2}u^2]^{3/2}} = \frac{q(1-v^2/c^2)}{2\epsilon_0} \left(\frac{c}{v}\right)^3 \int_{-1}^{1} \frac{du}{\left(\frac{c^2}{v^2}-1+u^2\right)^{3/2}}.$$

The integral is: $\displaystyle \left. \frac{u}{\left(\frac{c^2}{v^2}-1\right)\sqrt{\frac{c^2}{v^2}-1+u^2}} \right|_{-1}^{+1} = \frac{2}{\left(\frac{c^2}{v^2}-1\right)\frac{c}{v}} = \left(\frac{v}{c}\right)^3 \frac{2}{(1-v^2/c^2)}.$ So

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{q(1-v^2/c^2)}{2\epsilon_0} \left(\frac{c}{v}\right)^3 \left(\frac{v}{c}\right)^3 \frac{2}{(1-v^2/c^2)} = \boxed{\frac{q}{\epsilon_0}.} \quad \checkmark$$

So, our suspicion that Gauss's Law still applies was correct!

Q2) Rest mass of electron $m = 0.511\,\text{MeV}$

In S.I. units $m_{SI} = \frac{me}{c^2}$ kg,

where e is the electron charge.

<u>Falling under gravity</u>

$$V_{lost} = m_{SI}\,gh = \frac{me}{c^2}gh$$

$$t = \sqrt{\frac{2h}{g}}$$

<u>Radiated energy</u>

$$W = Pt = \frac{\mu_0 c\, q^2 \dot{\beta}^2}{6\pi}\sqrt{\frac{2h}{g}} = \frac{\mu_0 e^2}{6\pi c}\sqrt{2hg^3}$$

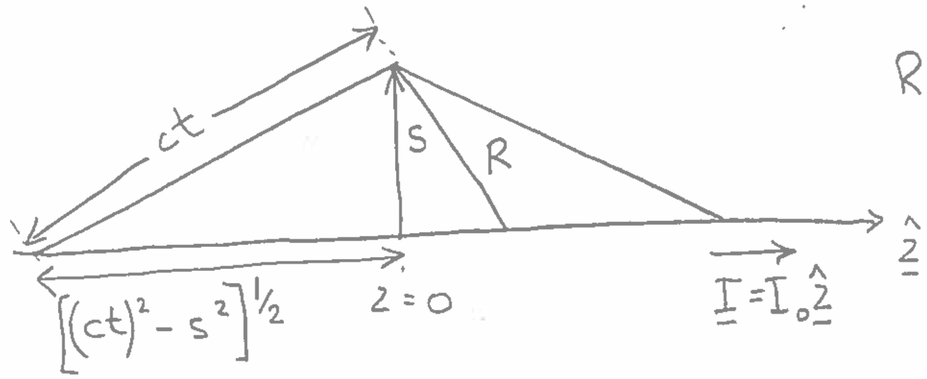
$$\frac{W}{V_{lost}} = \frac{\mu_0 e c}{6\pi m}\sqrt{\frac{2g}{h}}$$

$$= \frac{(4\pi\times10^{-7})(1.6\times10^{-19})(3\times10^{8})}{6\pi(0.511\times10^{6})}\sqrt{\frac{2(9.8)}{0.01}}$$

$$= 2.8\times10^{-22}.$$

Q3) At point P, a distance s from the wire, at time t after current becomes I_0:

only a limited section of the wire will contribute to $\underline{A}$ corresponding to $|z| < [(ct)^2 - s^2]^{1/2}$



$R = [s^2 + z^2]^{1/2}$

(a) Wire is uncharged : $V = 0$

Simplest choice for $\underline{A}$ is that only non-zero component is A_z

$$A_z(s,t) = \frac{\mu_0}{4\pi} \int_{-[(ct)^2-s^2]^{1/2}}^{[(ct)^2-s^2]^{1/2}} \frac{I_0}{R}\, dz = \frac{\mu_0 I_0}{2\pi} \int_0^{[(ct)^2-s^2]^{1/2}} [s^2+z^2]^{-1/2} dz$$

$$= \frac{\mu_0 I_0}{2\pi} \left[\ln\left\{ [s^2+z^2]^{1/2} + z \right\} \right]_0^{[(ct)^2-s^2]^{1/2}}$$

$$= \frac{\mu_0 I_0}{2\pi} \left[\ln\left\{ ct + [(ct)^2 - s^2]^{1/2} \right\} - \ln s \right]$$

$$\boxed{A_z = \frac{\mu_0 I_0}{2\pi} \ln K} \text{ , where } K = \frac{ct + [(ct)^2 - s^2]^{1/2}}{s}$$

23(b) $\underline{E} = -\nabla V - \dfrac{\partial \underline{A}}{\partial t}$

$\therefore$ Only non-zero component is E_z

$$E_z = -\frac{\partial A_z}{\partial t} = \frac{\mu_0 I_0}{2\pi}\frac{\partial}{\partial t}(\ln k)$$

$$= -\frac{\mu_0 I_0}{2\pi}\frac{1}{k}\left(c + \frac{1}{\cancel{2}}\frac{\cancel{2}c^2 t}{[(ct)^2 - s^2]^{1/2}}\right)\frac{1}{s}$$

$$= -\frac{\mu_0 I_0}{2\pi}\frac{1}{k}\,c\left(\frac{[(ct)^2 - s^2]^{1/2} + ct}{s}\right)\frac{1}{[(ct)^2 - s^2]^{1/2}}$$

$$\boxed{E_z = -\frac{\mu_0 I_0 c}{2\pi[(ct)^2 - s^2]^{1/2}}}$$

(c) $\underline{B} = \nabla \times \underline{A}$

Only non-zero component is A_z, which depends only on s.

$\therefore$ only non-zero component of $\nabla \times \underline{A}$ is

$$B_\phi = -\frac{\partial A_z}{\partial s} = -\frac{\mu_0 I_0}{2\pi}\frac{\partial}{\partial s}(\ln k)$$

Q3)(b) Contd.

$$\frac{\partial}{\partial s}(\ln K) = \frac{1}{K}\frac{\partial K}{\partial s}$$

$$= \frac{1}{K}\left[-\frac{K}{s} + \frac{1}{\cancel{2}}\frac{(-\cancel{2}\cancel{s})}{[(ct)^2-s^2]^{1/2}}\frac{1}{\cancel{s}}\right]$$

$$= \frac{-1}{Ks^2}\left[ct + [(ct)^2-s^2]^{1/2} + \frac{s^2}{[(ct)^2-s^2]^{1/2}}\right]$$

$$= \frac{-1}{Ks^2}\left[\frac{ct[(ct)^2-s^2]^{1/2} + (ct)^2 - \cancel{s^2} + \cancel{s^2}}{[(ct)^2-s^2]^{1/2}}\right]$$

$$= \frac{-ct}{Ks^2}\frac{[(ct)^2-s^2]^{1/2}+ct}{[(ct)^2-s^2]^{1/2}} = \frac{-ct}{\cancel{K}s^{\cancel{2}}}\frac{\cancel{K}\cancel{s}}{[(ct)^2-s^2]^{1/2}}$$

$$\therefore \quad B_\phi = \frac{\mu_0 I_0\, ct}{2\pi s[(ct)^2-s^2]^{1/2}}$$

As $t \to \infty$ $\quad B_\phi \to \dfrac{\mu_0 I_0}{2\pi s}$

That is, we recover the result for an infinite wire carrying current I_0 (as expected ;-)

Q3)
(d) $S = \frac{1}{\mu_0} \underline{E} \times \underline{B}$

Given $\underline{E} = E_z \hat{\underline{z}}$, $\underline{B} = B_\phi \hat{\underline{\phi}}$, only non-zero component of S is in the $\hat{\underline{s}}$ direction

$$\underline{S} = \mu_0 \left(\frac{I_0 c}{2\pi}\right)^2 \frac{t}{s((ct)^2 - s^2)} \hat{\underline{s}}$$

When $t = \frac{s}{c}$ (the time taken for signal traveling at speed c to carry information on change of current in wire to point P):

S diverges.

This is related to the (unphysical) acceleration of the charge carriers at $t = 0$ in the wire.

Q4)

Cross checking the total power radiated as bremsstrahlung requires us to evaluate the integral:

$$I = \int_{-1}^{1} \frac{(1-x^2)}{(1-\beta x)^5}\, dx \quad , \qquad \text{where} \quad x = \cos\theta$$

Integrating by parts with: $u = (1-x^2)$; $du = -2x\,dx$

$dv = (1-\beta x)^{-5} dx$; $v = \frac{1}{4\beta}(1-\beta x)^{-4}$

$$I = \underbrace{\left[(1-x^2)\frac{1}{4\beta}(1-\beta x)^{-4}\right]_{-1}^{1}}_{=0} - \left(\frac{-2}{4\beta}\right)\int_{-1}^{1} x(1-\beta x)^{-4} dx$$

Integrating by parts a second time with $dv = (1-\beta x)^{-4}$; $v = \frac{1}{3\beta}(1-\beta x)^{-3}$

$$I = \frac{2}{4\beta}\frac{1}{3\beta}\left\{\left[x(1-\beta x)^{-3}\right]_{-1}^{1} - \int_{-1}^{1}(1-\beta x)^{-3} dx\right\}$$

$$I = \frac{2}{4\beta}\frac{1}{3\beta}\left\{(1-\beta)^{-3} + (1+\beta)^{-3} - \frac{1}{2\beta}\underbrace{\left[(1-\beta x)^{-2}\right]_{-1}^{1}}\right\}$$

$$- \frac{1}{2\beta}\left[(1-\beta)^{-2} - (1+\beta)^{-2}\right]$$

$$= \frac{2}{4\beta}\frac{1}{3\beta}\frac{1}{2\beta(1-\beta)^3(1+\beta)^3}\left\{2\beta\left[(1+\beta)^3 + (1-\beta)^3\right] - (1-\beta^2)\left[(1+\beta)^2 - (1-\beta)^2\right]\right\}$$

$$= \frac{1}{12\beta^3}\frac{1}{(1-\beta^2)^3}\left\{2\beta\left[2+6\beta^2\right] - (1-\beta^2)\left[4\beta\right]\right\}$$

$$= \frac{\gamma^6}{12\beta^3}\left\{4\beta + 12\beta^3 - 4\beta + 4\beta^3\right\}$$

$$I = \frac{4}{3}\gamma^6 \quad \text{as required}$$

Q5)

Lecture 21

(a) According to [redacted], the maximum occurs at $\frac{d}{d\theta}\left[\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right] = 0$. Thus

$$\frac{2\sin\theta\cos\theta}{(1-\beta\cos\theta)5} - \frac{5\sin^2\theta(\beta\sin\theta)}{(1-\beta\cos\theta)^6} = 0 \Rightarrow 2\cos\theta(1-\beta\cos\theta) = 5\beta\sin^2\theta = 5\beta(1-\cos^2\theta);$$

$2\cos\theta - 2\beta\cos^2\theta = 5\beta - 5\beta\cos^2\theta$, or $3\beta\cos^2\theta + 2\cos\theta - 5\beta = 0$. So

$\cos\theta = \frac{-2\pm\sqrt{4+60\beta^2}}{6\beta} = \frac{1}{3\beta}\left(\pm\sqrt{1+15\beta^2}-1\right)$. We want the plus sign, since $\theta_m \to 90°(\cos\theta_m = 0)$ when $\beta \to 0$ ([redacted]): $\boxed{\theta_{\max} = \cos^{-1}\left(\frac{\sqrt{1+15\beta^2}-1}{3\beta}\right).}$

(b) For $v \approx c$, $\beta \approx 1$; write $\beta = 1-\epsilon$ (where $\epsilon \ll 1$), and expand to first order in ϵ:

$$\begin{aligned}\left(\frac{\sqrt{1+15\beta^2}-1}{3\beta}\right) &= \frac{1}{3(1-\epsilon)}\left[\sqrt{1+15(1-\epsilon)^2}-1\right] \cong \frac{1}{3}(1+\epsilon)\left[\sqrt{1+15(1-2\epsilon)}-1\right]\\ &= \frac{1}{3}(1+\epsilon)\left[\sqrt{16-30\epsilon}-1\right] = \frac{1}{3}(1+\epsilon)\left[4\sqrt{1-(15\epsilon/8)}-1\right] = \frac{1}{3}(1+\epsilon)\left[4\left(1-\frac{15}{16}\epsilon\right)-1\right]\\ &= \frac{1}{3}(1+\epsilon)\left(3-\frac{15}{4}\epsilon\right) = (1+\epsilon)(1-\frac{5}{4}\epsilon) \cong 1+\epsilon-\frac{5}{4}\epsilon = 1-\frac{1}{4}\epsilon.\end{aligned}$$

Evidently $\theta_{\max} \approx 0$, so $\cos\theta_{\max} \cong 1-\frac{1}{2}\theta_{\max}^2 = 1-\frac{1}{4}\epsilon \Rightarrow \theta_{\max}^2 = \frac{1}{2}\epsilon$, or $\theta_{\max} \cong \sqrt{\epsilon/2} = \boxed{\sqrt{(1-\beta)/2}.}$

(c) Let $f \equiv \frac{(dP/d\Omega|_{\theta_m})_{\text{ur}}}{(dP/d\Omega|_{\theta_m})_{\text{rest}}} = \left[\frac{\sin^2\theta_{\max}}{(1-\beta\cos\theta_{\max})^5}\right]_{\text{ur}}$. Now $\sin^2\theta_{\max} \cong \epsilon/2$, and

$(1-\beta\cos\theta_{\max}) \cong 1-(1-\epsilon)(1-\frac{1}{4}\epsilon) \cong 1-(1-\epsilon-\frac{1}{4}\epsilon) = \frac{5}{4}\epsilon$. So $f = \frac{\epsilon/2}{(5\epsilon/4)^5} = \left(\frac{4}{5}\right)^5\frac{1}{2\epsilon^4}$. But

$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(1-\epsilon)^2}} \cong \frac{1}{\sqrt{1-(1-2\epsilon)}} = \frac{1}{\sqrt{2\epsilon}} \Rightarrow \epsilon = \frac{1}{2\gamma^2}$. Therefore

$$f = \left(\frac{4}{5}\right)^5\frac{1}{2}(2\gamma^2)^4 = \boxed{\frac{1}{4}\left(\frac{8}{5}\right)^5\gamma^8 = 2.62\gamma^8.}$$

$\cos\theta \approx 1-\frac{\theta^2}{2}$