PHYS30441 Electrodynamics: Examples 3

1. In Lecture 14 we used the Lorentz transformation to work out the scalar component of the 4-potential produced by a point charge q in a frame of reference, S, in which it is moving along the x^1 axis with constant speed (in units of c) β:

$$\begin{aligned} A^0 &= \frac{V}{c} = \gamma A'^0 = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{1}{R'} \\ &= \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{1}{\left[\left(\gamma\left[x^1-\beta x^0\right]\right)^2+\left(x^2\right)^2+\left(x^3\right)^2\right]^{\frac{1}{2}}} \end{aligned}$$

Show that the potential A^0 may be expressed also as

$$A^0 = \frac{q}{4\pi\varepsilon_0 cR}\left(\frac{1}{1-\beta^2\sin^2\theta}\right)^{\frac{1}{2}},$$

where $\boldsymbol{R}$ is the vector from the current position $x^1=\beta x^0, x^2=0, x^3=0$ of the point charge to the point at which the field is evaluated. The magnitude of R is given by

$$R^2 = \left[x^1-\beta x^0\right]^2+\left(x^2\right)^2+\left(x^3\right)^2.$$

θ is the angle between $\boldsymbol{R}$ and the direction of motion.

2. (a) Show that $(\boldsymbol{E}\cdot\boldsymbol{B})$ is a Lorentz invariant by using the explicit transformation equations for the components of $\boldsymbol{E}$ and $\boldsymbol{B}$.

   (b) Show that $(E^2-c^2B^2)$ is a Lorentz invariant.
   Note: It would be interesting to do this in two different ways:

   i. By using the explicit transformation equations for $\boldsymbol{E}$ and $\boldsymbol{B}$.

   ii. *(Optional challenge)* The quantity $F_{\mu\nu}F^{\mu\nu}$ is manifestly a Lorentz invariant. (All Lorentz indices in the expression come in contravariant-covariant pairs, which are summed over.) By showing that

$$F_{\mu\nu}F^{\mu\nu} = \frac{-2}{c^2}\left(E^2-c^2B^2\right)$$

   we therefore can prove that $(E^2-c^2B^2)$ is a Lorentz invariant.

   Notes/hints on calculating $F_{\mu\nu}F^{\mu\nu}$:

   i. Because both 4-vector indices are contracted, $F_{\mu\nu}F^{\mu\nu}$ is a scalar. Because of the behaviour of $F_{\mu\nu}$ and $F^{\mu\nu}$ under Lorentz transformations, $F_{\mu\nu}F^{\mu\nu}$ is guaranteed to be a Lorentz scalar.

   ii. Show that the elements of $F_{\mu\nu}$ are the same as those of $F^{\mu\nu}$ except for the replacement $E_i \to -E_i$. (The signs of the B_i terms are unchanged.)

   iii. The contraction over the two 4-vector indices produces a sum containing 16 terms. It is, perhaps, simplest just to write out explicitly all 16 terms and add them together.
   [I hope you've noticed that you walk past an expression involving $F^{\mu\nu}F_{\mu\nu}$ in the entrance hall to the Schuster Building! (It appears as part of the "Lagrangian Density" of the Standard Model of particle physics[1])].

   (c) Suppose that in one inertial frame of reference $\boldsymbol{B}=0$ but $\boldsymbol{E}\neq 0$ (at some point P). Is it possible to find another frame in which the electric field is zero at P?

[1] Subtle advert for the 4th year courses on Field Theory and Gauge Theories ;–)