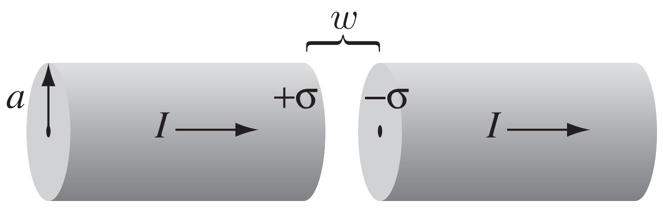
Electrodynamics (PHYS30441) Bonus Examples Sheet Terry Wyatt

1. A wire of radius a carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width w, forms a parallel-plate capacitor, as shown in the diagram. N.B. For the purposes of this example we shall be interested only in the fields within the gap. We shall assume that $w \ll a$ and, therefore, that the effects of any "fringe" fields for $r > a$ may be neglected.



   (a) Find the electric field $\boldsymbol{E}$ within the gap as a function of time t, assuming that $E = 0$ at $t = 0$.

   (b) Find the magnetic field $\boldsymbol{B}$ within the gap, at a distance $r < a$ from the axis. How does $\boldsymbol{B}$ in the gap compare to that within the current-carrying wire for $r < a$?

   (c) Find a vector potential $\boldsymbol{A}$ that is consistent with the $\boldsymbol{B}$ found above.

   (d) Find the energy density u and the Poynting vector $\boldsymbol{S}$ within the gap. Note especially the direction of $\boldsymbol{S}$. Using your results verify that the local conservation of electromagnetic energy is satisfied.

   (e) Draw a diagram showing the directions of the vectors $\boldsymbol{I}$, $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{A}$, $\boldsymbol{S}$.

2. In Lecture 3 we found the average potential over a sphere of radius r, centred on the origin, produced by a point charge q that is **outside** the sphere, at postition $\boldsymbol{r'} = z\hat{\boldsymbol{z}}$. (That is $z > r$).

   (a) Use an analogous approach to find the average potential over a sphere of radius r produced by a point charge q that is **inside** the sphere. That is, the same as in Lecture 3 except that $z < r$. In this case, of course, Laplace's equation does not hold within the sphere.

   (b) Show that, in general,
$$V_{\text{ave}} = V_{\text{centre}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 r},$$
   where V_{centre} is the potential at the centre due to all charges external to the sphere, and Q_{enc} is the total charge enclosed by the sphere.

3. A solid sphere of radius $\mathcal{K}$ is centered at the origin. The "northern" hemisphere carries a uniform charge density ρ_0, and the "southern" hemisphere a uniform charge density $-\rho_0$. Find the approximate electric field $\boldsymbol{E}(r, \theta)$ for points far from the sphere ($r \gg r'$).

4. A solid sphere of radius $\mathcal{K}$ is centered at the origin. It carries a uniform charge density ρ_0 and is rotating about the z axis with angular frequency ω.

   (a) What is the magnetic dipole moment of the sphere?

   (b) Find approximate expressions for the vector potential and the magnetic field at a point (r, θ) for $r \gg \mathcal{K}$.

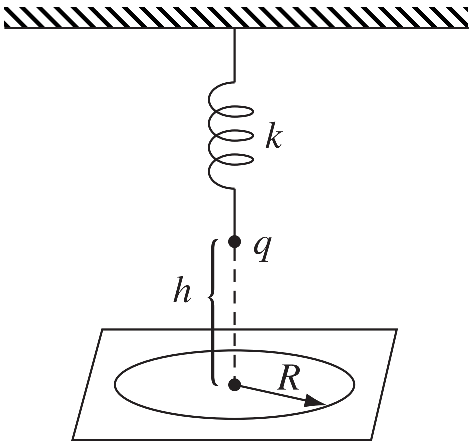
5. Show that the fractional energy lost per revolution by a highly relativistic charged particle in a circular orbit can be written as

$$\frac{\Delta T}{T} \rightarrow k\frac{T^3}{R},$$

where T is the relativistic kinetic energy, R is the radius of the orbit and k is a constant. Evaluate the fractional energy loss per revolution for an electron with $T = 2$ GeV and $R = 5$ m.

6. In a classical model of the Hydrogen atom, a (non-relativistic) electron moves in a circular orbit around the proton. If the radius were initially equal to the Bohr radius a_0, estimate the initial rate of energy loss in eV/sec. Describe the subsequent behaviour of the classical system.

7. A particle of mass m and charge q is attached to a spring with force constant k, hanging from the ceiling (as shown in the figure). Its equilibrium position is a distance h above the floor. It is pulled down a distance d below equilibrium and released, at time $t = 0$. Assume non-relativistic motion (!)



(a) Show that the intensity of the radiation hitting the floor, as a function of the distance R from the point directly below q is given by

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}A} \right\rangle = \frac{\mu_0 q^2 d^2 \omega^4 R^2 h}{32\pi^2 c \left(R^2 + h^2\right)^{5/2}},$$

where $\omega^2 = \dfrac{k}{m}$.

[Note: The intensity here is the average power P per unit area A of floor.]

(b) At what value of R is the radiation most intense?

(c) As a check on your formula, assume the floor is of infinite extent, and calculate the average energy per unit time striking the entire floor. Is it what you would expect?

You may make use of the result that $\displaystyle\int_0^\infty \frac{R^3}{\left(R^2 + h^2\right)^{5/2}}\, \mathrm{d}R = \frac{2}{3h}$.

(d) Because it is losing energy in the form of radiation, the amplitude of the oscillation will gradually decrease. After what time τ has the amplitude been reduced to d/e?

[Note: Assume the fraction of the total energy lost in one cycle is very small.]

8. In an inertial frame S an electromagnetic plane wave of angular frequency ω is traveling in the x direction through the vacuum. It is polarized such that the electric field is in the y direction. The amplitude of the electric field is E_0.

(a) Write down an expression for the electric field $\boldsymbol{E}(x, y, z, t)$. [Be sure to define any auxiliary quantities you introduce, in terms of ω, E_0, and the constants of nature.]

(b) Using the relevant Maxwell Equation, or otherwise, write down a (vector) expression for the associated magnetic field $\boldsymbol{B}(x, y, z, t)$.

(c) This same wave is observed from an inertial system S', which is moving in the x direction with speed (in units of c) β relative to the original frame S. Find the electric and magnetic fields in S', and express them in terms of the S' coordinates: $\boldsymbol{E}'(x', y', z', t')$ and $\boldsymbol{B}'(x', y', z', t')$. [Again, be sure to define any auxiliary quantities you introduce.]

(d) What is the angular frequency ω' of the wave in S'? Interpret this result. What is the (angular) wavenumber k' of the wave in S'? From ω' and k', determine the speed of the waves in S'. Is it what you expected?

(e) What is the ratio of the intensity in S' to the intensity in S? What happens to the amplitude, frequency, and intensity of the wave in S', as $\beta \to 1$?

9. Consider the invariant mass, $m_{1,2,3}$, of a system of three particles, $i = 1, 2, 3$. If all three particles are *ultra-relativistic*, show that $m_{1,2,3}$ may be expressed as

$$m_{1,2,3}^2 = m_{1,2}^2 + m_{1,3}^2 + m_{2,3}^2,$$

where $m_{i,j}$ is the invariant mass of the system of two particles, i, j.

[Note: If you work directly in terms of the 4-vectors $\underset{\sim}{p_i}$ *the proof is very short. It is possible to write out the invariant mass quite straightforwardly in terms of the separate energies and momenta of the three particles, but this involves rather more algebra.]*

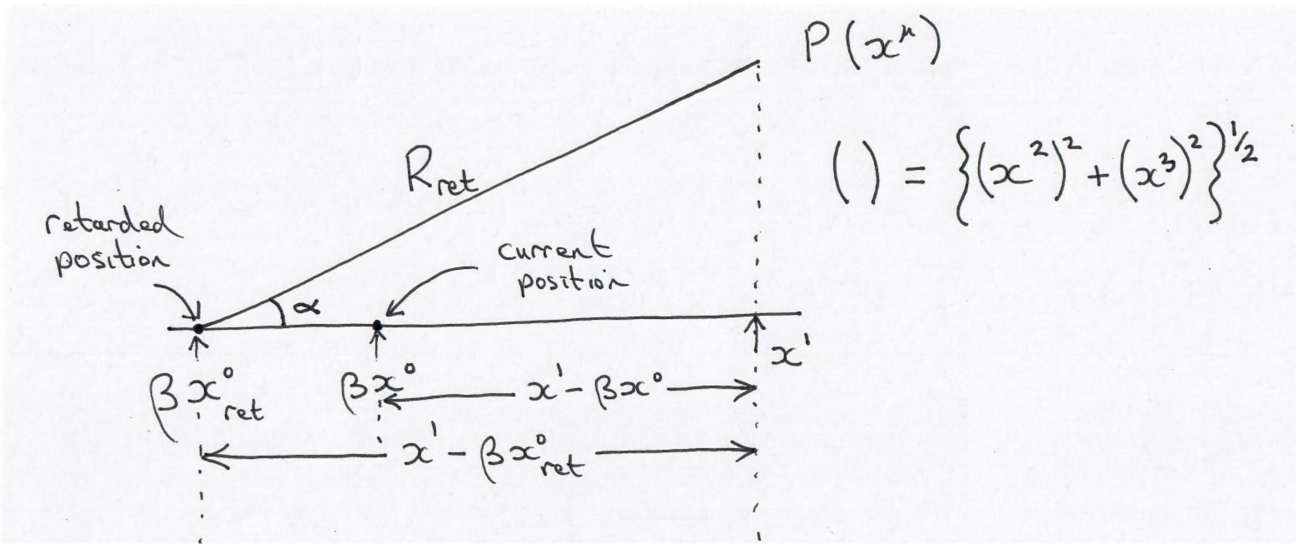
10. In the Lecture 14 we used the Lorentz transformations to work out the scalar potential produced by a point charge in a frame of reference, S, in which it is moving with constant velocity (in units of c) β in the x^1 direction:

$$\begin{aligned} A^0 &= \frac{V}{c} = \gamma A'^0 = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{1}{R'} \\ &= \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{1}{\left[(\gamma\left[x^1-\beta x^0\right])^2+(x^2)^2+(x^3)^2\right]^{\frac{1}{2}}} \end{aligned}$$

From this we derived, in Lecture 19a, the alternative "Liénard-Wiechert" formulation of the scalar potential for a point particle of charge q moving with velocity $\boldsymbol{\beta}$:

$$A^0 = \frac{V}{c} = \frac{1}{4\pi\varepsilon_0 c}\frac{q}{\left[R\left(1-\boldsymbol{\beta}\cdot\widehat{\boldsymbol{R}}\right)\right]_{\text{ret}}} = \frac{1}{4\pi\varepsilon_0 c}\frac{q}{\left[R-\boldsymbol{\beta}\cdot\boldsymbol{R}\right]_{\text{ret}}}$$

In this problem we shall work in the other direction: starting with the Liénard-Wiechert potentials and "deriving" the expressions from Lecture 14.



Let's start by examinig the diagram, which indicates the "current" position of the charge at time x^0, the position of the charge at the "retarded" time x^0_{ret}, and the vector $\boldsymbol{R}_{\text{ret}}$ from the retarded position to the point $P(x^\mu)$ at which the potentials and fields are to be evaluated.

(a) Find an expression that relates R_{ret} to the distance travelled by the charge between the retarded time x^0_{ret} and the "current" time x^0.

(b) Thus find expressions for R_{ret} and $\cos\alpha$ in terms of β and the components of x^μ.

(c) Substitute these expressions into that given above for the Liénard-Wiechert scalar potential and compare the result with that obtained in Lecture 14 (also given above).

Looking this way around at the potentials at $P(x^\mu)$ emphasizes the fact that physically they originate from the retarded time and position x^μ_{ret}. Perhaps this helps to emphasize how surprising it is that the potentials are centred on the "current" space-time point x^0?

11. The accelerating RF cavities in the LHC deliver a maximum electric field of 2 MV/m. At the maximum beam energy of 7 TeV what is the radiated power for a proton being accelerated in this field?

12. In Lecture 14 we showed that at most one point on the trajectory of a particle moving with $v < c$ communicates with the potential/field measurement point $P(x, y, z)$ at any given observation time t. In some cases there may be no such point. (That is, an observer at $P(x, y, z)$ would not see the particle at time t). As an example, consider a particle in so-called "hyperbolic" motion along the x axis, whose position is given by:

$$\boldsymbol{w}(t) = \sqrt{b^2 + (ct)^2}\ \widehat{\boldsymbol{x}}.$$

Sketch the path of the particle in the x, ct plane. (Pick, say, the negative value for the square root.) At a few representative points along the path, draw the trajectory of a light signal emitted by the particle at that point. What region on your graph corresponds to positions and times (x, ct) from which the particle cannot be seen? At what time does someone at the point $+x$ first see the particle? (Prior to this the potential at x is zero.) Is it possible for a particle, once seen, to disappear from view?

13. A particle of mass m and charge q is attached to a spring with force constant k. It is pulled a distance d below equilibrium and released from rest.

(a) Find an expression for the total energy radiated by the particle as a function of the displacement x from the equilibrium position.

(b) Integrating with respect to x find the total energy radiated over one complete cycle of the oscillation of the particle.

You may find the following integral useful: $\displaystyle\int_0^d \frac{x^2\,dx}{(d^2 - x^2)^{\frac{1}{2}}} = \frac{\pi d^2}{4}.$

(c) Thinking back to Question 7, can you think of a way to cross-check your answer?
Hint: Starting (i) from your answer to part (b) of this question, and (ii) from your answer to part (a) of Question 7, can you show that the time-averaged radiated power is given by

$$\langle P \rangle = \frac{\mu_0 q^2 k^2 d^2}{12\pi m^2 c}.$$

(d) If the time-averaged radiated power in the approximate rest frame of the oscillating charge is $\langle P \rangle$, what is the radiated power in a frame in which the oscillating charge is moving with speed (in units of c) β

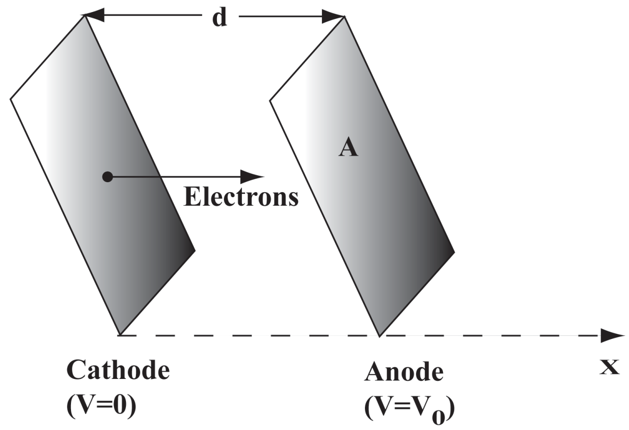
i. along the x axis;

ii. in a direction perpendicular to the x axis.

(e) *Optional:* Integrating by parts and looking up some "standard" integrals can you prove the result

$$\int_0^d \frac{x^2\,\mathrm{d}x}{(d^2 - x^2)^{\frac{1}{2}}} = \frac{\pi d^2}{4}\ ?$$

14. Consider the figure below in which electrons leave the cathode at $x = 0$ at potential $V = 0$ and reach the anode at $x = d$, held at a potential $V = V_0$.



The cloud of moving electrons within the gap is referred to as 'space charge' and this builds up rapidly to the point where it reduces the field at the surface of the cathode to zero. Thereafter, a steady current per unit area, $\boldsymbol{j}$, flows. Assuming that the plates are much larger than their relative separation (in order that edge fringe-field effects can be safely ignored) then V, ρ (the charge density) and v (the speed of the electrons) are all functions of x only.

(a) Write down Poisson's equation for the region between the plates.

(b) Assuming the electrons start from rest at the cathode, what is their speed at point x, where the potential is $V(x)$?

(c) In the steady state the current per unit area, $\boldsymbol{j}$, is independent of x. In this case what is the relation between ρ and v?

(d) Use these results to obtain a differential equation for V by elmininating ρ and v.

(e) Solve this equation to obtain V as a function of x, V_0 and d. Plot $V(x)$ and compare it to that expected without the effect of space charge. Also, find ρ and v as functions of x.

[*Hint:* One way to do this is to just make a guess for the general functional dependence of V on x and to check whether or not your guess works out by evaluating $\mathrm{d}^2V/\mathrm{d}x^2$ for your guess.]

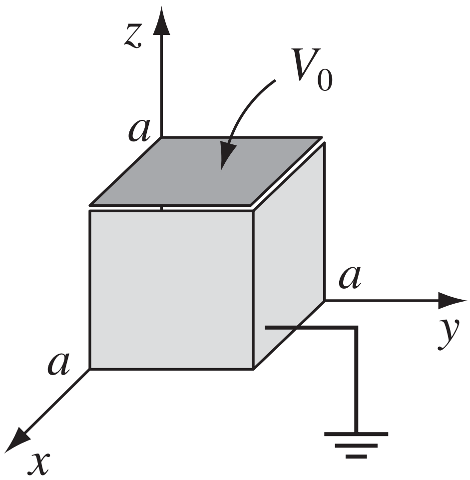
(f) Show that

$$j = KV_0^{3/2}$$

and find the constant K.

[*Note:* This equation describes the important *Child-Langmuir* law for emission of electrons from an electrode. It is valid for other geometries as well as the plane plates pictured. Notice the current and voltage are non-linearly related. That is, this is not an Ohm's law relation.]

15. As shown in the diagram, a cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (as shown in the figure). The top, which lies in the plane $z = a$, is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. (That, is V_0 is independent of x and y.)



(a) Find the potential inside the box.

(b) By inspection, can you work out what the potential at the centre of the cube $(a/2,\ a/2,\ a/2)$ should be?

(c) Consider the line $x = a/2, y = a/2$. Sketch the behaviour of $V(a/2, a/2, z)$ and $\boldsymbol{E}(a/2, a/2, z)$ as a function of z along this line, indicating key features on your sketch.