PHYS30441

Electrodynamics

Bonus Example Sheet

Solutions — Terry Wyatt.

Q1) (a) From Gauss's Law $E = \frac{\sigma}{\varepsilon_0}$, where σ is the surface charge density.

$$\frac{d\sigma}{dt} = \frac{I}{\pi a^2} \quad \therefore \quad \sigma = \frac{It}{\pi a^2} \quad \left(\text{since } \sigma = 0 \text{ at } t = 0\right)$$

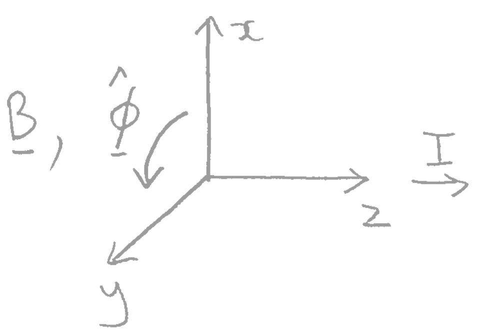
$$\therefore \quad E = \frac{It}{\pi a^2 \varepsilon_0}$$

(b) $$2\pi r B = \oint \underline{B} \cdot d\underline{l} = \int (\nabla \times \underline{B}) \cdot d\underline{a} = \int \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \cdot d\underline{a}$$

$$= \mu_0 \varepsilon_0 \pi r^2 \frac{I}{\pi a^2 \varepsilon_0}$$

$$\therefore \quad B = \frac{\mu_0 I}{2\pi a^2} r \,,$$

which is exactly the same value it has within the wire.



Let I be in the $\hat{\underline{z}}$ direction

By right hand rule $\underline{B}$ is in the $\hat{\underline{\phi}}$ direction.

(c) Since $\underline{I}$ is $I\,\hat{\underline{z}}$ it is natural to assume that $\underline{A} = A_z \hat{\underline{z}}$

$$\frac{\mu_0 I}{2\pi a^2} r \,\hat{\underline{\phi}} = \underline{B} = \nabla \times \underline{A} = -\frac{\partial A_z}{\partial r}\,\hat{\underline{\phi}}$$

$$\therefore \quad A_z = -\frac{\mu_0 I r^2}{4\pi a^2}$$

would be a possible choice.

(d) Consider a cylinder co-axial with the wire of unit length and radius r.

Let U be energy contained in this cylinder

$$\frac{dU}{dt} = \pi r^2 \frac{du}{dt},$$

where $u = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$ is the energy density in the e.m. fields.

$$\frac{du}{dt} = \frac{\varepsilon_0}{2} \cdot 2 \cdot E \frac{dE}{dt} + 0$$

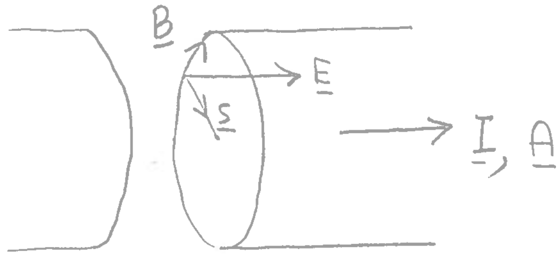
↑ since B is constant.

$$= \varepsilon_0 \left(\frac{I}{\pi a^2 \varepsilon_0} \right)^2 t$$

$$\therefore \quad \frac{dU}{dt} = \pi r^2 \varepsilon_0 \left(\frac{I}{\pi a^2 \varepsilon_0} \right)^2 t = \frac{I^2 t r^2}{\varepsilon_0 \pi a^4} \quad \text{(Eqn A)}$$

The Poynting vector

$$\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B} \quad \text{radially inwards}$$



$$= \frac{1}{\mu_0} \left(\frac{I}{\pi a^2} \right) \frac{t}{\varepsilon_0} \left(\frac{I}{\pi a^2} \right) \frac{\mu_0 r}{2}$$

Flux of S into the cylindrical surface

$$\int \underline{S} \cdot \underline{da} = 2\pi r S = \frac{I^2 t r^2}{\varepsilon_0 \pi a^4} = \frac{dU}{dt}$$

(from Eqn A)

which corresponds to conservation of energy at all values of r.

(e) See diagram above.

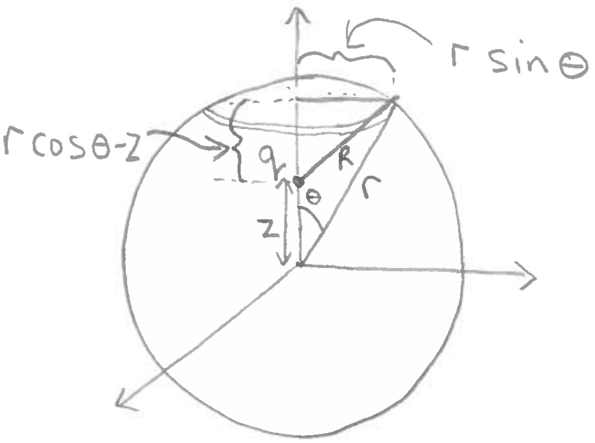
Q2)

(a) If the charge were at the centre of the sphere then trivially

$$\langle V \rangle = V_{constant} = \frac{q}{4\pi\varepsilon_0 r},$$

but let us consider the more general case of an off-centre charge.

Without loss of generality let us consider the position $\underline{r}' = (x, y, z) = (0, 0, z)$



$$R = \left[r^2 + z^2 - 2rz\cos\theta\right]^{1/2}$$

$$\langle V \rangle = \frac{\int V d\Omega}{\int d\Omega} = \frac{\frac{q}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} \frac{r^2 \sin\theta \, d\theta}{\left[r^2+z^2-2rz\cos\theta\right]^{1/2}}}{4\pi r^2}$$

Use the substitution $y = \cos\theta$, $dy = -\sin\theta \, d\theta$

$$\langle V \rangle = \frac{q}{4\pi\varepsilon_0} \frac{r^2}{4\pi r^2} 2\pi \left(-\int_1^{-1} \frac{dy}{\left[r^2+z^2-2rzy\right]^{1/2}} \right)$$

$$= \frac{q}{8\pi\varepsilon_0} \left[\frac{1}{rz} \left(r^2+z^2-2rzy \right)^{1/2} \right]_1^{-1}$$

$$= \frac{q}{8\pi\varepsilon_0} \cdot \frac{1}{rz} \left[(r+z) - (r-z) \right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}$$

That is, the contribution to $\langle V \rangle$ is independent of the distance z to the centre of the sphere of the location of the charge.

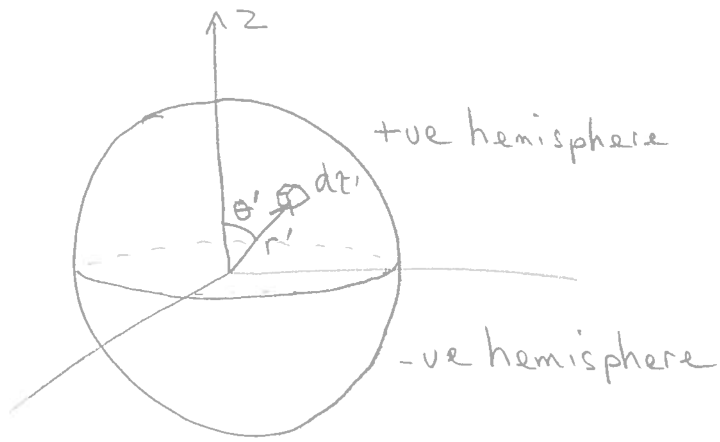
(b) For a collection of charges q_i inside the sphere the contribution to $\langle V \rangle$ will therefore be given by

$\dfrac{Q_{enc}}{4\pi\varepsilon_0 r}$ where $Q_{enc} = \sum_i q_i$ the total charge

In Lecture 3 we showed that the contribution to $\langle V \rangle$ over the sphere produced by a charge outside the sphere is equal to V_{centre}, the potential at the centre of the sphere produced by the charge.

Therefore, in total $\langle V \rangle = V_{centre} + \dfrac{Q_{enc}}{4\pi\varepsilon_0 r}$

Q3)



$$z' = r'\cos\theta'$$

Total charge is zero.

$\therefore$ Dominant term is dipole

Dipole moment $\underline{p} = p\,\hat{\underline{z}}$ by symmetry

$$p = \underbrace{\int z'\rho\, d\tau'}_{\text{over sphere}} = 2\underbrace{\int z'\rho\, d\tau'}_{\substack{\text{over}\\ \text{northern}\\ \text{hemisphere}}} \quad , \quad \text{where } d\tau' = (r')^2 \sin\theta'\, dr' d\phi' d\theta'$$

$\uparrow$

By symmetry since for southern hemisphere $\rho \to -\rho$ and $z' \to -z'$.

$$\therefore \; p = 2\rho_0 \int_0^{2\pi} d\phi' \int_0^{K} (r')^3\, dr' \int_0^{\pi/2} \cos\theta' \sin\theta'\, d\theta'$$

$$= 2\rho_0 \cdot 2\pi \cdot \frac{K^4}{4} \cdot \left[\frac{\sin^2\theta'}{2}\right]_0^{\pi/2} \quad \left(\text{since } d(\sin\theta') = \cos\theta' d\theta'\right)$$

$$= \frac{\pi \rho_0 K^4}{2}$$

$$\therefore \; E_{dipole} = \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right] = \frac{\rho_0 K^4}{8\varepsilon_0 r^3}\left[2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right]$$

Q4)

(a) In Additional Examples 2, Q5) (a) we showed that for a spherical shell with surface charge density σ and radius K rotating with $\underline{\omega} = \omega \hat{\underline{z}}$

$$\underline{m} = \frac{4}{3}\pi \sigma \omega K^4$$

$\therefore$ For spherical shell radius r', thickness dr' and volume charge density ρ_0 we have

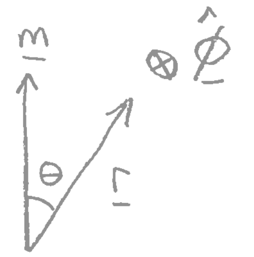
$$m = \frac{4}{3}\pi \rho_0 \omega (r')^4 dr' \qquad \text{since } \sigma = \rho_0 dr'$$

Integrating to obtain solid sphere

$$\underline{m} = \left[\frac{4}{3}\pi \rho_0 \omega \int_0^K (r')^4 dr'\right]\hat{\underline{z}} = \frac{4}{15}\pi \rho_0 \omega K^5 \hat{\underline{z}}$$

Q4)

(b)



$$\underline{m} \times \hat{\underline{r}} = m \sin\theta \, \hat{\underline{\phi}}$$

$$\therefore \underline{A}_{dipole} = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \underline{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi} \frac{4}{15} \pi \rho_0 \omega K^5 \frac{\sin\theta}{r^2} \hat{\underline{\phi}}$$

$$= \frac{\mu_0 \rho_0 \omega K^5}{15} \frac{\sin\theta}{r^2} \hat{\underline{\phi}}$$

Evaluate $\underline{B}_{dipole}$ either by using

i) $\underline{B}_{dipole} = \nabla \times \underline{A}_{dipole}$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right)\right]\hat{\underline{r}} - \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r A_\phi\right)\right]\hat{\underline{\theta}}$$

$$= \frac{\mu_0 \rho_0 \omega K^5}{15}\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right)\right]\hat{\underline{r}} - \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\right]\hat{\underline{\theta}}\right\}$$

$$= \frac{\mu_0 \rho_0 \omega K^5}{15}\left\{\frac{2\cos\theta\sin\theta}{r^3\sin\theta}\hat{\underline{r}} - \left(-\frac{\sin\theta}{r^3}\right)\hat{\underline{\theta}}\right\}$$

$$= \frac{\mu_0 \rho_0 \omega K^5}{15 r^3}\left\{2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right\}$$

or alternatively using the expression from the Lectures

(ii) $\underline{B}_{dipole} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)$

$$= \frac{\mu_0}{4\pi r^3}\,\frac{4\pi}{15}\rho_0 \omega R^5\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)$$

$$= \frac{\mu_0 \rho_0 \omega R^5}{15 r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)$$

Q5)

(5) With $\underline{v}$ and $\underline{a}$ $\perp$

$$P = \frac{q^2}{6\pi\varepsilon_0 c}\dot{\beta}^2\gamma^4 \quad \text{with } \beta \sim 1$$

Circular orbit: $a = \frac{v^2}{R} \simeq \frac{c^2}{R}$

So $\dot{\beta} \simeq \frac{c}{R}$

Kinetic energy $T = (\gamma - 1)Mc^2$ M: rest mass.

So $\gamma \to \frac{T}{Mc^2}$

in one period of orbit:

$$\Delta t = \frac{2\pi R}{v} \simeq \frac{2\pi R}{c}$$

So energy loss / revolution

$$\Delta T = \frac{q^2}{6\pi\varepsilon_0 c}\left(\frac{c}{R}\right)^2\left(\frac{T}{Mc^2}\right)^4\left(\frac{2\pi R}{c}\right)$$

So $\frac{\Delta T}{T} = k\frac{T^3}{R}$ with $k = \frac{q^2}{3\varepsilon_0 (Mc^2)^4}$

with $T \simeq 2\text{ GeV}$, $M_e c^2 = 0.5\times 10^{-3}\text{ GeV}$ $R = 5\text{m}$

$$\frac{\Delta T}{T} \simeq 10^{-4}$$

Q6)

b) $F = \frac{m_e v^2}{r} = \frac{e^2}{4\pi\varepsilon_0 a_0^2}$; $\gamma = 1$

So $a = \frac{v^2}{r} = \frac{e^2}{4\pi\varepsilon_0 m_e a_0^2}$

$\beta \ll 1$ use $P_L = \frac{e^2}{6\pi\varepsilon_0 c^3}\left(\frac{e^2}{4\pi\varepsilon_0 m_e a_0^2}\right)^2$

with $a_0 = 0.5\,\text{Å} = 0.5\times10^{-10}\,m$ $m_e c^2 = 0.5\times10^6\,eV$

$P \simeq 2.9\times10^{11}\,eV\,s^{-1}$

Taking Binding energy as typical $\Delta E \sim 10\,eV$

$\Delta t \sim \frac{\Delta E}{P} \sim 3.5\times10^{-11}\,s$ or 35 ps.

Q7) (a) Two ways of approaching this problem yield the same starting point

(i) Treat moving charge using a multipole expansion about the centre point of the oscillation

- constant monopole term (irrelevant for radiation)
- oscillating dipole term $p = p_0 \cos \omega t$, with $p_0 = q d$ and $\omega^2 = k/m$

In Lecture 23, electric dipole radiation given as:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 p_0^2 \omega^4}{32 \pi^2 c} \sin^2 \theta$$

$$= \frac{\mu_0 q^2 d^2 \omega^4}{32 \pi^2 c} \sin^2 \theta$$

(ii) Treat as Larmor radiation (since $v \ll c$) from an accelerating point charge

From Lecture 20:

$$\frac{dP}{d\Omega} = \frac{\mu_0 c q^2}{16\pi^2} \sin^2\theta \, \dot{\beta}^2$$

Let $x = d\cos\omega t$, where $\omega^2 = k/m$

$a = -d\omega^2 \cos\omega t$

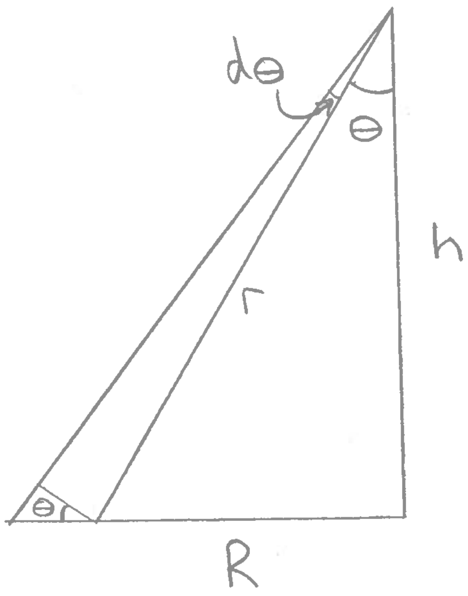
$$\therefore \langle \dot{\beta}^2 \rangle = \frac{\langle a^2 \rangle}{c^2} = \frac{d^2\omega^4}{2c^2} \,, \quad \text{since } \langle \cos^2\omega t \rangle = 1/2$$

$$\therefore \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c} \sin^2\theta \,, \quad \text{as in (a)(i) above.}$$

(Note: I think this is quite a neat short-cut derivation of the formula for dipole radiation!

cf. e.g., Griffiths Section 11.1, where the derivation "from first principles" using retarded potentials extends over several pages.)

We are asked to calculate $\left\langle \frac{dP}{dA} \right\rangle$



Let $r^2 = R^2 + h^2$

$\sin\theta = \frac{R}{r}$

$\cos\theta = \frac{h}{r}$

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

$$dA = \frac{r^2 \sin\theta \, d\theta \, d\phi}{\cos\theta} = \frac{r^2 \, d\Omega}{\cos\theta}$$

$$\left\langle \frac{dP}{dA} \right\rangle = \frac{\cos\theta}{r^2} \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c} \frac{\cos\theta \sin^2\theta}{r^2}$$

$$= \frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c} \frac{R^2 h}{(R^2+h^2)^{5/2}}$$

Q7) (b)

$$\frac{d}{dR}\left\langle \frac{dP}{dA} \right\rangle \propto \frac{2R}{(R^2+h^2)^{5/2}} - \frac{5}{2} \cdot \frac{2R}{(R^2+h^2)^{7/2}} \cdot R^2 = 0 \text{ at maximum}$$

$$\therefore (R^2+h^2) - \frac{5R^2}{2} = 0$$

$$\therefore R^2 = \frac{2h^2}{3} \quad \text{at maximum.}$$

(c) Integrating over $dA = 2\pi R\, dR$

$$\langle P \rangle_{floor} = \frac{\mu_0 q^2 d^2 \omega^4 h}{32\pi^2 c} 2\pi \int_0^\infty \frac{R^3\, dR}{(R^2+h^2)^{5/2}}$$

$$= \frac{\mu_0 q^2 d^2 \omega^4}{24\pi c}$$, using the integral given in the question.

This is half of the value given in Lecture 23 for an oscillating dipole

$$\langle P \rangle_{dipole} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$, which makes sense since the other half is radiated upwards.

Following discussion of (a)(ii) this result is also consistent with $P = \frac{\mu_0 c q^2 \dot\beta^2}{6\pi}$ for Larmor radiation given in Lecture 20.

Q7) (d)

At $t=0$ energy of oscillator $U(0) = \frac{1}{2} k d^2$

When amplitude $= \frac{d}{e}$

$$U(\tau) = U(0) e^{-2}$$

Rate of energy loss due to radiation

$$-\frac{dU}{dt} = 2\langle P\rangle_{floor} = \frac{\mu_0 q^2 d^2 \omega^4}{12\pi c}$$

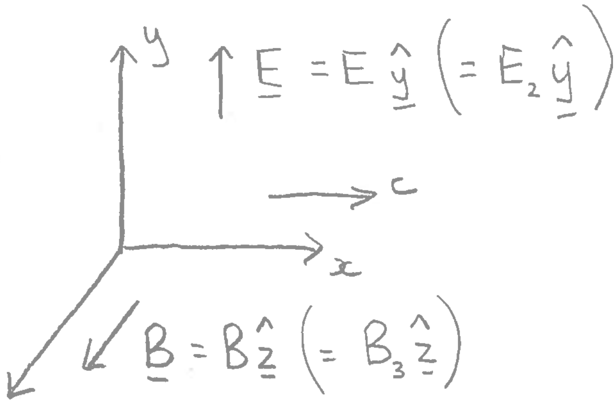
$$= \frac{\mu_0 q^2 \omega^4}{12\pi c} \frac{2U}{k}$$

$$\therefore \quad U(t) = U(0) e^{-\left(\frac{\mu_0 q^2 \omega^4}{6\pi c k}\right)t}$$

$$\therefore \quad -2 = -\frac{\mu_0 q^2 \omega^4}{6\pi c k} \tau$$

$$\therefore \quad \tau = \frac{12\pi c k}{\mu_0 q^2 \omega^4} \quad \text{or} \quad \frac{12\pi c m^2}{\mu_0 q^2 k}$$

Q8)



(a) Let $\underline{E} = E \cos(\omega t - kx)\hat{\underline{y}}$, where $\frac{\omega}{k} = c$

(b) Using $\nabla \times \underline{B} = \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t}$ {must have non-zero component only in $\hat{\underline{y}}$ }

$$(\nabla \times B)_y = \cancel{\frac{\partial B_x}{\partial z}} - \frac{\partial B_z}{\partial x}$$

$$\therefore \; -\frac{\partial B}{\partial x} = \frac{1}{c^2}\frac{\partial E}{\partial t} = -\frac{\omega E \sin(\omega t - kx)}{c^2}$$

$$\therefore \; B = \frac{\omega}{k}\frac{1}{c^2} E \cos(\omega t - kx)$$

or $\underline{B} = \frac{E}{c}\cos(\omega t - kx)\,\hat{\underline{z}}$

(or $B_3 = \frac{E_2}{c}$)

Q8) (c) Using the transformation equations (fields)

$$\frac{E_2'}{c} = \gamma\left(\frac{E_2}{c} - \beta B_3\right) = \gamma(1-\beta)\frac{E_2}{c}$$

$$B_3' = \gamma\left(B_3 - \beta \frac{E_2}{c}\right) = \gamma(1-\beta)\frac{E_2}{c} \quad \left(= \gamma(1-\beta)B_3\right)$$

Using the inverse transformation equations for coordinates

$$x = \gamma(x' + \beta ct) \quad , \quad ct = \gamma(ct' + \beta x')$$

allows us to re-write

$$\cos[\omega t - kx] \Rightarrow \cos\left[\omega\gamma\left(t' + \frac{\beta}{c}x'\right) - k\gamma(x' + \beta ct')\right]$$

$$= \cos\left[\gamma(\omega - kc\beta)t' - \gamma\left(k - \frac{\omega}{c}\beta\right)x'\right]$$

$$= \cos\left[\gamma(1-\beta)\omega t' - \gamma(1-\beta)k t'\right] \quad \left(\text{using } \frac{\omega}{k} = c\right)$$

$$= \cos\left[\omega' t' - k' x'\right]$$

where we have written

$$\omega' = \gamma(1-\beta)\omega = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}\omega$$

$$k' = \gamma(1-\beta)k = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}k$$

Q8)

(d) $\therefore$ ω and k decrease (and thus λ increases) by the same factor $\left(\frac{1-\beta}{1+\beta}\right)$.

This corresponds to the "relativistic Doppler shift"

Of course, the apparent speed of the photons of light in S' $\frac{\omega'}{k'} = \frac{\omega}{k} = c$ is unchanged!

Q8)

(e) Intensity of light in S is given by the Poynting vector

$$S = \frac{1}{\mu_0} \left| \underline{E} \times \underline{B} \right| = \frac{1}{\mu_0 c} E^2$$

In frame S' Poynting vector becomes

$$S' = \gamma^2 (1-\beta)^2 S = \frac{1-\beta}{1+\beta} S$$

As $\beta \to 1$

$S \to 0$, amplitude $\to 0$

$\omega \to 0$, $\lambda \to \infty$

Light becomes more and more redshifted.

Q9)

Working in terms of the 4-vectors $\underset{\sim}{p}_i$, $i = 1,2,3$, we can write (using $c=1$ for brevity)

$$m_{1,2,3}^2 = \left(\underset{\sim}{p}_1 + \underset{\sim}{p}_2 + \underset{\sim}{p}_3\right)^2$$

$$= \underset{\sim}{p}_1^2 + \underset{\sim}{p}_2^2 + \underset{\sim}{p}_3^2 + 2\left(\underset{\sim}{p}_1 \cdot \underset{\sim}{p}_2 + \underset{\sim}{p}_1 \cdot \underset{\sim}{p}_2 + \underset{\sim}{p}_2 \cdot \underset{\sim}{p}_3\right)$$

$$m_{1,2}^2 + m_{1,3}^2 + m_{2,3}^2 = \left(\underset{\sim}{p}_1 + \underset{\sim}{p}_2\right)^2 + \left(\underset{\sim}{p}_1 + \underset{\sim}{p}_3\right)^2 + \left(\underset{\sim}{p}_2 + \underset{\sim}{p}_3\right)^2$$

$$= \underset{\sim}{p}_1^2 + \underset{\sim}{p}_2^2 + 2\underset{\sim}{p}_1 \cdot \underset{\sim}{p}_2$$

$$+ \underset{\sim}{p}_1^2 + \underset{\sim}{p}_3^2 + 2\underset{\sim}{p}_1 \cdot \underset{\sim}{p}_3$$

$$+ \underset{\sim}{p}_2^2 + \underset{\sim}{p}_3^2 + 2\underset{\sim}{p}_2 \cdot \underset{\sim}{p}_3$$

If particle i is ultra-relativistic then $m_i \ll E_i \approx p_i$ and since $\underset{\sim}{p}_i^2 = m_i^2$

Q10) (a) In the time interval $x^0 - x^0_{ret} = R_{ret}$ the charge travels a distance $\beta(x^0 - x^0_{ret}) = \beta R_{ret}$

(b) Eliminate x^0_{ret} from expressions for R_{ret} and $\cos\alpha$:

$$\cos\alpha = \frac{\beta R_{ret} + (x^1 - \beta x^0)}{R_{ret}} \qquad (1)$$

and by pythagoras:

$$\left[\beta R_{ret} + (x^1 - \beta x^0)\right]^2 + (\ )^2 = R^2_{ret}$$

$$R^2_{ret}(1-\beta^2) - 2\beta(x^1 - \beta x^0)R_{ret} - \{(x^1 - \beta x^0)^2 + (\ )^2\} = 0$$

$$\therefore \quad R_{ret} = \frac{2\beta(x^1 - \beta x^0) \pm \sqrt{4\beta^2(x^1-\beta x^0)^2 + 4(1-\beta^2)\{(x^1-\beta x^0)^2 + (\ )^2\}}}{2(1-\beta^2)}$$

Choose + square root to give $R_{ret} > 0$

$$R_{ret} = \frac{\beta(x^1 - \beta x^0) + \sqrt{(x^1 - \beta x^0)^2 + (1-\beta^2)(\ )^2}}{(1-\beta^2)} \qquad (2)$$

(c) Using eqn (1)

$$R_{ret} - \underline{\beta} \cdot \underline{R}_{ret} = R_{ret} - \beta R_{ret} \cos\alpha = R_{ret} - \beta\left(\beta R_{ret} + (x^1 - \beta x^0)\right)$$

$$= (1-\beta^2) R_{ret} - \beta(x^1 - \beta x^0)$$

$$= \sqrt{(x^1 - \beta x^0)^2 + (1-\beta^2)(\ )^2} = \gamma^{-1}\sqrt{\gamma^2(x^1 - \beta x^0)^2 + (\ )^2} \qquad (3)$$

↑ using (2)

Subsitute (3) into expression for L-W potential

$$A^0 = \frac{q}{4\pi\varepsilon_0 c}\,\gamma\,\frac{1}{\left[\gamma^2(x^1 - \beta x^0)^2 + (\ )^2\right]^{1/2}}$$ as required.

Q 11) For $\underline{\dot{\beta}} \parallel \underline{\beta}$

$$\frac{dp}{dt} = \frac{d}{dt}\left(\gamma m c \beta\right) = mc\left(\underset{\substack{\uparrow \\ \text{non-zero for } \underline{\dot{\beta}} \parallel \underline{\beta}\ !}}{\dot{\gamma}} \beta + \gamma \dot{\beta}\right)$$

$$\dot{\gamma} = \frac{d}{dt}\left(\left[1-\beta^2\right]^{-1/2}\right) = -\frac{1}{2}\left[1-\beta^2\right]^{-3/2}\left(-2\beta\dot{\beta}\right) = \gamma^3 \beta\dot{\beta}$$

$$\frac{dp}{dt} = mc\gamma^3\dot{\beta}\underbrace{\left(\beta^2 + \frac{1}{\gamma^2}\right)}_{\beta^2 + 1 - \beta^2 = 1} = mc\gamma^3\dot{\beta} = qE$$

$$P_{\parallel} = \frac{\mu_0 c q^2 \gamma^6 \dot{\beta}^2}{6\pi} = \frac{\mu_0 c q^2 \cancel{\gamma^6}}{6\pi} \frac{q^2 E^2}{(mc)^2 \cancel{\gamma^6}}$$

$$\boxed{P_{\parallel} = \frac{\mu_0 c q^4 E^2}{6\pi (mc)^2}}$$

(remove one power of e to give result in eV)

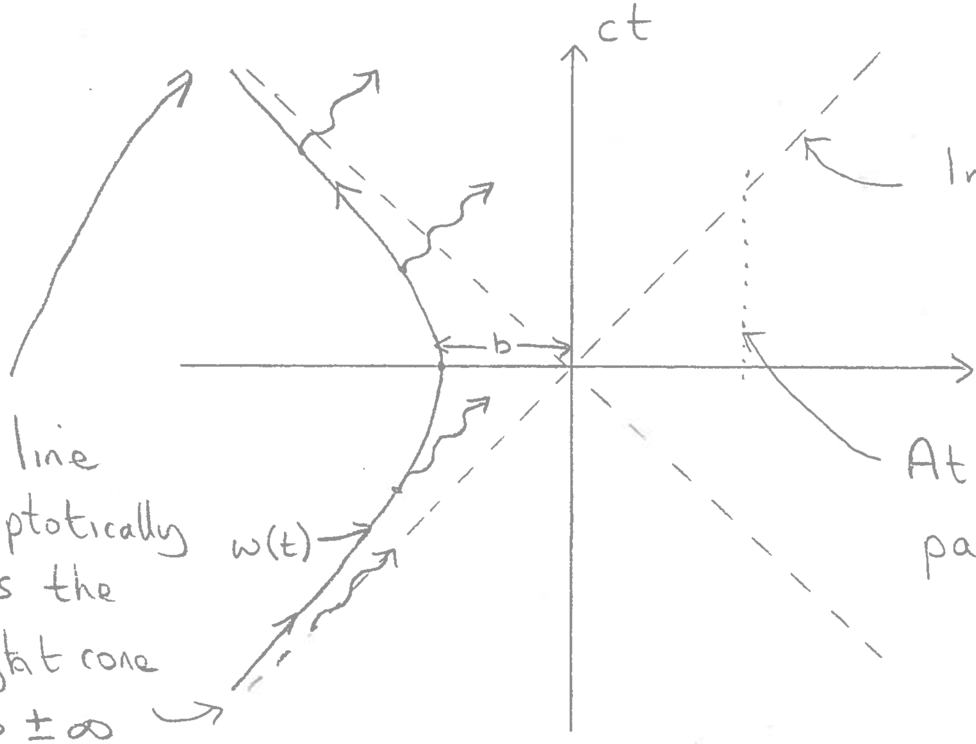
↓

$$P_{\parallel} = \frac{\left(4\pi \times 10^{-7}\right) \times \left(3 \times 10^8\right)^3 \times \left(1.6 \times 10^{-19}\right)^3 \times \left(2 \times 10^6\right)^2}{6\pi \times \underbrace{\left(938 \times 10^6\right)^2 \times \left(1.6 \times 10^{-19}\right)^2}_{(mc^2)^2 \text{ for proton}}}$$

$\approx 10^{-6}$ eV/s i.e. very small!

compared to the energy of the protons!

Q12)



In the (x, t) region below this line the particle cannot be seen at the space-time point $x = 0$, $t = 0$.

At position $+x$ the earliest the particle is observed is time $t = \frac{x}{c}$.

The world line $w(t)$ asymptotically approaches the drawn light cone at $t \to \pm\infty$

Once observed at a given position x the particle cannot "disappear."

As $t \to \infty$ $\quad w(t) \to ct$

$$\frac{dw}{dt} = \frac{1}{\cancel{2}} \frac{\cancel{2} c^2 t}{\left[b^2 + (ct)^2\right]^{1/2}}$$

As $t \to \infty$ $\quad \frac{dw}{dt} \to c$

Q 13)

(a)

$$a = \frac{F}{m} = -\frac{kx}{m}$$

$$\therefore \dot{\beta} = -\frac{kx}{mc}$$

Larmor radiation, total radiated power

$$P = \frac{\mu_0 c q^2}{6\pi} \dot{\beta}^2$$

$$= \frac{\mu_0 q^2 k^2 x^2}{6\pi m^2 c}$$

Another useful result from conservation of energy

$$\frac{1}{2} m v^2 = \frac{1}{2} k \left(d^2 - x^2\right)$$

$$\therefore v = \sqrt{\frac{k}{m}\left(d^2 - x^2\right)}$$

will be needed in part (b)

(b) Total energy radiated over one complete oscillation

$$\mathcal{E} = \int_0^{\tau} P \, dt$$

where $\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$ is the period.

Since the speed $v = \frac{dx}{dt}$, $dt = \frac{dx}{v}$

and we can write

$$\mathcal{E} = 4\int_0^{d} \frac{P}{v} \, dx$$

, where the factor of 4 accounts for the fact that $x = 0 \rightarrow x = d$ is $1/4$ of a complete oscillation.

Substituting in from answer to (a):

$$\mathcal{E} = 4 \, \frac{\mu_0 q^2 k^2}{6\pi m^2 c}\sqrt{\frac{m}{k}} \underbrace{\int_0^{d} \frac{x^2}{(d^2 - x^2)^{1/2}} \, dx}_{\frac{\pi d^2}{4}}$$

$$= \frac{\mu_0 q^2 k^2 d^2}{16 m^2 c}\sqrt{\frac{m}{k}}$$

(c) Cross-check against Q7) answers

Average power radiated $\langle P \rangle = \dfrac{E}{\tau} = \dfrac{\mu_0 q^2 k^2 d^2}{12\pi m^2 c}$

In question 7(a) we obtained

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c} \sin^2\theta .$$

Using $\omega^4 = \dfrac{k^2}{m^2}$ and $\displaystyle\int \sin^2\theta \, d\Omega = \frac{8\pi}{3}$ $\left\{ \begin{array}{l} \text{see} \\ \text{Lecture 20} \end{array} \right\}$

$$\langle P \rangle = \int \left\langle \frac{dP}{d\Omega} \right\rangle d\Omega = \frac{\mu_0 q^2 k^2 d^2}{12\pi m^2 c} ,$$

as obtained above.

(d) Because the radiation in the rest frame of the oscillator has zero total momentum, the time averaged <P> is a Lorentz invariant. Therefore, in both cases we obtain the same <P> as in the answer to part (c).

This perhaps surprising result can be understood as follows:

(i) In the frame in which the oscillator is moving along the x axis the acceleration is reduced by the sixth power of gamma. This is the same power of gamma by which the radiated power is increased in the formula for bremsstrahlung, and so all the factors of gamma cancel.

(ii) In the frame in which the oscillator is moving perpendicular to the x axis the fourth powers of gamma cancel simlarly.

(e) Using $du = \dfrac{dx}{(d^2-x^2)^{1/2}}$

$$u = \int \frac{dx}{(d^2-x^2)^{1/2}} = \arcsin\left(\frac{x}{d}\right) + c$$

$$v = x^2$$

$$dv = 2x\,dx$$

we can write the required integral as

$$I = \int_0^d v\,du = \big[uv\big]_0^d - \int_0^d u\,dv$$

$$= \left[\arcsin\left(\frac{x}{d}\right) x^2\right]_0^d - 2\int_0^d x\left[\arcsin\left(\frac{x}{d}\right)\right] dx$$

Using the following "standard" integral and $\arcsin(1) = \pi/2$

$$\int x \arcsin(ax)\,dx = \frac{x^2\arcsin(ax)}{2} - \frac{\arcsin(ax)}{4\,a^2} + \frac{x\sqrt{1-a^2x^2}}{4\,a} + C$$

we can write

$$I = \left[\frac{\pi}{2}d^2 - 2\left\{\frac{d^2}{2}\frac{\pi}{2} - \frac{\pi}{2}\frac{d^2}{4} + 0\right\}\right] - \left[0 - 0 + 0\right]$$

$$= \pi d^2\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{4}\right] = \frac{\pi d^2}{4}, \text{ as required.}$$

Question 14)

(a) Since V is a function only of x Poisson's equation becomes

$$\nabla^2 V = \frac{d^2 V}{dx^2} = \frac{-\rho}{\varepsilon_0}$$

(b) At $x=0$ Potential Energy + Kinetic Energy $= 0$

$(qV = -eV = 0)$ $\left(\frac{1}{2}mv^2 = 0\right)$

$\therefore$ At x $-eV(x) + \frac{1}{2}mv^2 = 0$

$$v = \sqrt{\frac{2eV(x)}{m}}$$

(c) $j = \rho v$ (−ve since ρ is −ve)

(d) $\rho = \frac{j}{v} = j\sqrt{\frac{m}{2eV}}$

$$\frac{d^2V}{dx^2} = -\frac{j}{\varepsilon_0}\sqrt{\frac{m}{2eV}}$$

(e) Let $V = C x^{a}$ $\quad \therefore \quad \dfrac{1}{V^{1/2}} = \dfrac{1}{C^{1/2} x^{a/2}}$

Second constant of integration $= 0$ since $V = 0$ at $x = 0$.

$$\frac{d^2V}{dx^2} = C\,a(a-1)\,x^{a-2} \quad \propto \quad \frac{1}{x^{a/2}}$$

Using the answer to part (d) which gives $\dfrac{d^2V}{dx^2} \propto \dfrac{1}{V^{1/2}}$

Equating powers of x on the two sides of this equation gives

$$a - 2 = -\frac{a}{2} \quad \Rightarrow \quad a = 4/3$$

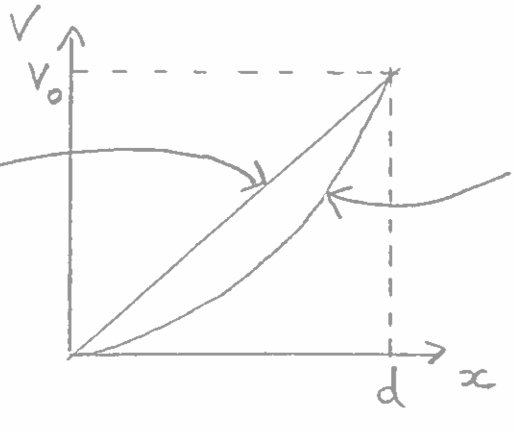
The boundary condition that $V = V_0$ at $x = d$ gives

$$V_0 = C\,d^{4/3} \qquad \therefore \quad V = V_0 \frac{x^{4/3}}{d^{4/3}}$$

$$v = \sqrt{\frac{2eV}{m}} = \sqrt{\frac{2eV_0}{m}}\;\frac{x^{2/3}}{d^{2/3}}$$

$$\rho = -\varepsilon_0 \frac{d^2V}{dx^2} = -\varepsilon_0\,\frac{4}{9}\,\frac{V_0}{d^{4/3}}\,\frac{1}{x^{2/3}}$$

$V = V_0 \dfrac{x}{d}$ (without space charge)



$V = V_0 \dfrac{x^{4/3}}{d^{4/3}}$ (with space charge)

(f) $j = \rho v$

$$= \left(-\varepsilon_0 \frac{4}{9} \frac{V_0}{d^{4/3}} \frac{1}{x^{2/3}}\right)\left(\sqrt{\frac{2eV_0}{m}} \frac{x^{2/3}}{d^{2/3}}\right)$$

$$= \left(-\varepsilon_0 \frac{4}{9d^2}\sqrt{\frac{2e}{m}}\right) V_0^{3/2}$$

This is of the required form with K equal to the expression within the parentheses.

Q15) (a)

Using separation of variables as in lecture 4 we can write the three dimensional Laplace's equation in the form

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

$$(-\alpha^2) + (-\beta^2) + (\alpha^2+\beta^2) = 0$$

Where α and β are constants and the signs are motivated by the need to achieve sin/cos solutions for x and y to satisfy the boundary conditions.

Solutions of the form :

$$X(x) = A\cos\alpha x + B\sin\alpha x$$

$$Y(y) = C\cos\beta y + D\sin\beta y$$

$$Z(z) = E e^{(\alpha^2+\beta^2)^{1/2} z} + F e^{-(\alpha^2+\beta^2)^{1/2} z}$$

Apply the boundary conditions

$x = 0, V = 0$: $A = 0$

$x = a, V = 0$: $\alpha = \frac{n\pi}{a}$ $\quad n = 1, 2, 3, \ldots$

$y = 0, V = 0$: $C = 0$

$y = a, V = 0$: $\beta = \frac{m\pi}{a}$ $\quad m = 1, 2, 3, \ldots$

$z = 0, V = 0$: $F = -E$

$$\therefore Z(z) = E\left(e^{(\alpha^2+\beta^2)^{1/2} z} - e^{-(\alpha^2+\beta^2)^{1/2} z}\right)$$

$$= 2E\sinh\left[\frac{\pi}{a}(n^2+m^2)^{1/2} z\right]$$

Putting it all together and combining constants

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\frac{\pi}{a}\left[n^2+m^2\right]^{1/2} z\right)$$

Evaluate C_{nm} by imposing final boundary condition

$z = a$, $V = V_0$ and using Fourier analysis

$$V_0 = \sum_n \sum_m C_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\pi\left[n^2+m^2\right]^{1/2}\right)$$

Using standard results:

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) dx = \frac{a}{2}\delta_{nn'}$$

$$\frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right) dx = \frac{2}{n\pi}\left(1-\cos n\pi\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{n\pi} & \text{if } n \text{ odd.} \end{cases}$$

$$\therefore C_{nm} \sinh\left(\pi\left[n^2+m^2\right]^{1/2}\right)$$

$$= \left(\frac{2}{a}\right)^2 V_0 \int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dx\,dy = \begin{cases} 0 & \text{if } n \text{ or } m \text{ even} \\ \frac{16 V_0}{\pi^2 nm} & \text{if } n \text{ and } m \text{ both odd.} \end{cases}$$

Which gives finally :

$$V(x,y,z) = \frac{16V_0}{\pi^2} \underbrace{\sum_n \sum_m}_{\substack{n\ \&\ m \\ \text{odd integers}}} \left[\frac{1}{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \frac{\sinh\left(\frac{\pi}{a}\left[n^2+m^2\right]^{1/2} z\right)}{\sinh\left(\pi\left[n^2+m^2\right]^{1/2}\right)} \right.$$

(b) Consider cube in which all <u>six</u> sides are maintained at V_0. By symmetry centre of cube would also be at potential V_0.

Since V is a <u>scalar</u>, by symmetry + superposition principle, each side must contribute $V_0/6$ to the potential at the centre of the cube.

(c)

$$E_z = -\frac{\partial V}{\partial z} = \sum_{n} \sum_{m} \frac{1}{nm}\frac{\pi}{a}\left[n^2+m^2\right]^{1/2} \frac{\cosh\left(\frac{\pi}{a}\left[n^2+m^2\right]^{1/2} z\right)}{\sinh\left(\pi\left[n^2+m^2\right]^{1/2}\right)}$$

odd integers

By symmetry $E_x = E_y = 0$ along $x = y = a/2$!