PHYS30441 Electrodynamics: Revision Examples Class - Solutions

1. (a) Using index notation:

$$\nabla \times (\mathbf{r} \times \mathbf{B}) = \widehat{\boldsymbol{x}}_{\boldsymbol{i}}\, \epsilon_{ijk} \frac{\partial}{\partial x_j} [\mathbf{r} \times \mathbf{B}]_k = \widehat{\boldsymbol{x}}_{\boldsymbol{i}}\, \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{kmn} x_m B_n = \widehat{\boldsymbol{x}}_{\boldsymbol{i}}\, (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\, \delta_{jm} B_n,$$

since

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}, \qquad \frac{\partial x_m}{\partial x_j} = \delta_{jm}, \qquad \frac{\partial B_n}{\partial x_j} = 0.$$

$$\therefore \nabla \times (\mathbf{r} \times \mathbf{B}) = \widehat{\boldsymbol{x}}_{\boldsymbol{i}}\, \delta_{in}\, (1 - 3)\, B_n = -2\widehat{\boldsymbol{x}}_{\boldsymbol{i}}\, B_i = -2\mathbf{B}.$$

since

$$\delta_{im}\delta_{jn}\delta_{jm} = \delta_{im}\delta_{mn} = \delta_{in}, \qquad \delta_{in}\delta_{jm}\delta_{jm} = \delta_{in}\delta_{mm} = 3\delta_{in}.$$

$$\therefore \nabla \times \mathbf{A} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{B}) = \mathbf{B}.$$

Alternative answer, using the given identity:

$$\nabla \times \mathbf{A} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{B}) = -\frac{1}{2}\left[\mathbf{r}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{r}) + (\mathbf{B} \cdot \nabla)\mathbf{r} - (\mathbf{r} \cdot \nabla)\mathbf{B}\right].$$

The magnetic field is uniform, and hence the first and fourth terms are zero. In Cartesian coordinates, the second term is

$$\mathbf{B}(\nabla \cdot \mathbf{r}) = \mathbf{B}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3\mathbf{B}.$$

The third term is

$$(\mathbf{B} \cdot \nabla)\mathbf{r} = \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right)(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = \mathbf{B}.$$

$$\Rightarrow \nabla \times \mathbf{A} = -\frac{1}{2}(-3\mathbf{B} + \mathbf{B}) = \mathbf{B}$$

as required.

(b)

$$\nabla \cdot \mathbf{A} = -\frac{1}{2}\nabla \cdot (\mathbf{r} \times \mathbf{B}) = -\frac{1}{2}\left[\mathbf{B} \cdot (\nabla \times \mathbf{r}) - \mathbf{r} \cdot (\nabla \times \mathbf{B})\right],$$

using vector identity (6), so the Coulomb gauge condition is satisfied, since $\nabla \times \mathbf{r} = 0$ and $\nabla \times \mathbf{B} = 0$.

This can also be shown very simply using index notation:

$$\nabla \cdot \mathbf{A} \propto \nabla \cdot (\mathbf{r} \times \mathbf{B}) = \frac{\partial}{\partial x_i}\left[\epsilon_{ijk} x_j B_k\right] = \delta_{ij}\epsilon_{ijk} B_k = 0,$$

since $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$. δ_{ij} is non-zero only if $i = j$ and ϵ_{ijk} is non-zero only if $i \neq j$.

2. Consider the disk to consist of a large number of thin rings. Consider a single ring of inner radius r and width dr. The charge on such a ring is $dq = 2\pi r\sigma dr$. Since the charge is rotating, the moving charge corresponds to a current

$$dI = \frac{dq}{dt} = \frac{2\pi r\sigma dr}{2\pi/\omega} = \sigma\omega r dr.$$

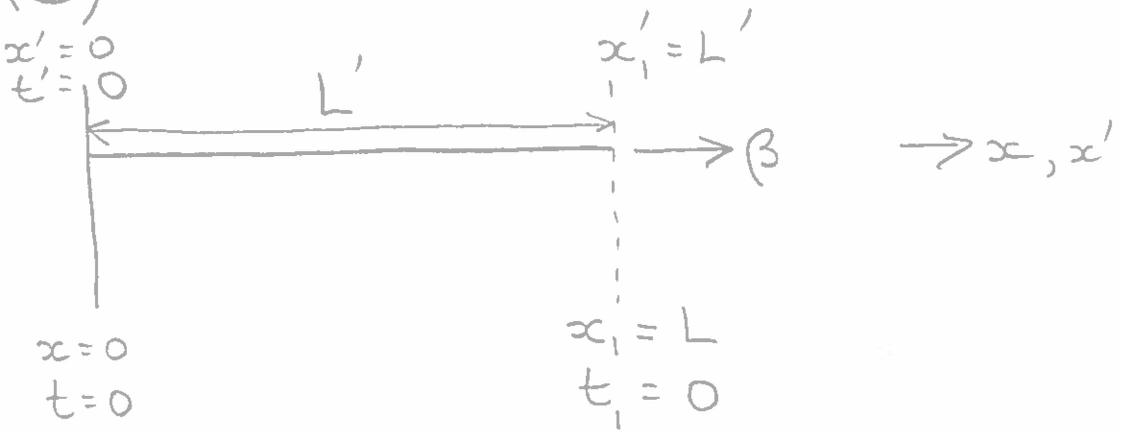
The dipole moment of this ring is therefore equal to

$$d\mathbf{m} = (\pi r^2)dI\hat{\mathbf{z}}.$$

(i.e. area $\times$ current, directed perpendicular to the plane of the ring). Hence, the total dipole moment of the disk is equal to

$$\mathbf{m} = \int_0^R \pi\sigma\omega r^3 dr\hat{\mathbf{z}} = \frac{\pi}{4}\sigma\omega R^4\hat{\mathbf{z}}.$$

Q3)



(a) Lorentz transformation :

$$x_1' = \gamma(x_1 - \beta c t_1)$$

$$L' = \gamma x_1 - 0$$

$$\therefore x_1 = \frac{L'}{\gamma}$$

(b) Simultaneous (in frame S) measurements of the positions of the two ends of the rod allow us to determine the length, L, of the rod in S.

$$\therefore x_1 - 0 = L = L'/\gamma$$

The rod is 'Lorentz contracted' in the frame S, in which it is moving.

(c) Lorentz transformation:

$$ct_1' = \gamma(ct_1 - \beta x_1) = 0 - \gamma\beta x_1$$

$$\therefore\ t_1' = -\frac{\beta L'}{c}$$

So, although in frame S the two measurements are made simultaneously, in frame S' this is not the case.

(d) Lorentz transformation:

$$x_1 = \gamma(x_1' + \beta c t') = \gamma x_1' + 0,$$ since

in order to measure a length in frame S' we have to make measurements simutaneous in S' (at, say, $t' = 0$)

$$\therefore\ x_1' = \frac{x_1}{\gamma} = \frac{L'}{\gamma^2}$$