PHYS30441 Electrodynamics: Examples Class 2

In these problems we shall practice using the "Minkowski Representation" or index notation for 4-vectors in Special Relativity that we introduced in Lecture 11.

1. In this first problem we shall revise the transformation of velocities from one frame to another that you first met in Quantum Physics and Relativity (PHYS10121). Velocity is a 3-dimensional vector, but does not directly constitute the vector part of a 4-vector. Therefore velocity does not transform directly according to the Lorentz transformation. However, we can use the Lorentz transformation to work out how the components of velocity do transform from one frame to another.

   Let us consider an inertial frame of reference S' to be moving with speed[1] (in units of c) β_{LT} along the x^1 axis relative to frame S.

   (a) Consider a particle that in frame S' is moving along the x'^2 direction with speed (in units of c) β'^2. That is,

   $$\beta'^2 = \frac{\Delta x'^2}{c\Delta t'} = \frac{\Delta x'^2}{\Delta x'^0} \quad \text{and} \quad \beta'^1 = \beta'^3 = 0.$$

   Using the inverse Lorentz transformations[2]

   $$\begin{aligned} \Delta x^0 &= \gamma_{\mathrm{LT}} \left(\Delta x'^0 + \beta_{\mathrm{LT}} \Delta x'^1\right) \\ \Delta x^1 &= \gamma_{\mathrm{LT}} \left(\beta_{\mathrm{LT}} \Delta x'^0 + \Delta x'^1\right) \\ \Delta x^2 &= \Delta x'^2 \\ \Delta x^3 &= \Delta x'^3, \end{aligned}$$

   find an expression for the velocity of the particle $\boldsymbol{\beta} = (\beta^1, \beta^2, \beta^3)$ in frame S.

   Can you come up with an "intuitive" explanation/understanding of this result?

   (b) Consider a particle that in frame S' is moving along the x'^1 direction with speed (in units of c) β'^1. That is,

   $$\beta'^1 = \frac{\Delta x'^1}{c\Delta t'} = \frac{\Delta x'^1}{\Delta x'^0} \quad \text{and} \quad \beta'^2 = \beta'^3 = 0.$$

   Find an expression for the velocity of the particle $\boldsymbol{\beta} = (\beta^1, \beta^2, \beta^3)$ in frame S.

2. *(More challenging) This topic is not covered in PHYS10121, but is very important for our understanding of Electrodynamics.*

   When we come to consider the radiation by accelerating charged particles we shall need to transform accelerations from the rest frame of a particle into a frame in which it is moving at high speeds.

   Again, let us consider an inertial frame of reference S' to be moving with speed (in units of c) β_{LT} along the x^1 axis relative to frame S.

   (a) Consider a particle that in frame S' has an acceleration $\dot{\beta}'^2$ along the x'^2 direction. That is,

   $$\dot{\beta}'^2 = \frac{\mathrm{d}}{\mathrm{d}t'}(\beta'^2) = \frac{\Delta(\beta'^2)}{\Delta t'} = \frac{c\Delta(\beta'^2)}{\Delta x'^0}, \quad \dot{\beta}'^1 = \dot{\beta}'^3 = 0 \quad \text{and} \quad \beta'^1 = \beta'^3 = 0.$$

[1] We use the symbol β_{LT} here just to avoid any possible confusion between the speed of relative motion between the two frames and the speed of a particle within any particular frame.

[2] Again, just to avoid any possible confusion, note that γ_{LT} here arises from the relative motion between the two frames of reference, and so would be expressed in terms of β_{LT}.

Using the result from Q1 (a) and the inverse Lorentz transformations find an expression for the acceleration $\dot{\beta}^2$ of the particle in frame S.

Can you come up with an "intuitive" explanation/understanding of this result?

(b) Consider a particle that in frame S' has an acceleration $\dot{\beta}'^1$ along the x'^1 direction. That is,

$$\dot{\beta}'^1 = \frac{\mathrm{d}}{\mathrm{d}t'}(\beta'^1) = \frac{\Delta(\beta'^1)}{\Delta t'} = \frac{c\Delta(\beta'^1)}{\Delta x'^0} \quad \text{and} \quad \dot{\beta}'^2 = \dot{\beta}'^3 = 0.$$

Using the result from Q1 (b) and the inverse Lorentz transformations find an expression for the acceleration $\dot{\beta}^1$ of the particle in frame S.

How does your expression for $\dot{\beta}^1$ simplify for the case that the particle is approximately at rest (moving with a speed much less than the speed of light) in frame S'?

Thinking back to the concepts of "time dilation" and "length contraction" that you explored in PHYS10121, can you come up with an "intuitive" explanation/understanding of this simplified result?

3. Suppose magnetic monopoles of magnetic charge g were to exist.

(a) Use a modified version of Maxwell's equation ($\nabla.\boldsymbol{B} = \mu_0\rho_m$) to find the magnetic field $\boldsymbol{B}$ around a point monopole ($\rho_m(\boldsymbol{r}) = g\delta^3(\boldsymbol{r})$).

(b) Verify that a possible choice of the vector potential of this field is

$$\boldsymbol{A} = \frac{\mu_0 g}{4\pi}\frac{(1-\cos\theta)}{r\sin\theta}\hat{\boldsymbol{\phi}}.$$