PHYS30441

Electrodynamics

Example Sheet 2

Solutions – Terry Wyatt.

Q1) (a) $$\beta^2 = \frac{\Delta x^2}{\Delta x^0} = \frac{\Delta x'^2}{\gamma_{LT}(\Delta x'^0 + \underbrace{\beta_{LT}\Delta x'^1}_{})}$$

$= 0$ since $\beta'^1 = 0$ and so $\Delta x'^1 = 0$

$$\therefore \boxed{\beta^2 = \frac{\beta'^2}{\gamma_{LT}} \text{ and } \underline{\beta} = \left(\beta_{LT}, \frac{\beta'^2}{\gamma_{LT}}, 0\right)}$$

Same distance travelled in S, S' but in a shorter time in S'.

(b) $$\beta^1 = \frac{\Delta x^1}{\Delta x^0} = \frac{\gamma_{LT}(\beta_{LT}\Delta x'^0 + \Delta x'^1)}{\gamma_{LT}(\Delta x'^0 + \beta_{LT}\Delta x'^1)}$$

$$= \frac{\left(\beta_{LT} + \frac{\Delta x'^1}{\Delta x'^0}\right)}{\left(1 + \beta_{LT}\frac{\Delta x'^1}{\Delta x'^0}\right)} = \frac{\beta_{LT} + \beta'^1}{1 + \beta_{LT}\beta'^1}$$

(Relative motion of particle in S')

(Relative motion of S' in S.)

$$\beta^2 = \beta^3 = 0$$

This is useful practice to help us prepare for doing the accelerations!

Optional alternative answer to Question 1

As an interesting alternative approach to Q1 we can make use of the 4-velocity

$$u^\mu = \gamma(c, \underline{v}) = \gamma c(1, \underline{\beta})$$

1(a) $$u'^\mu = c\gamma_2 \begin{pmatrix} 1 \\ 0 \\ \beta'^2 \\ 0 \end{pmatrix},$$ where $\gamma_2 = \left[1 - (\beta'^2)^2\right]^{-1/2}$ arises from motion of particle in frame S'

Since u'^μ is a 4-vector we can use the (inverse) Lorentz transformation directly

$$u^\mu = (\Lambda^{-1})^\mu{}_\nu u'^\nu = \begin{pmatrix} \gamma_{LT} & \gamma_{LT}\beta_{LT} & 0 & \\ \gamma_{LT}\beta_{LT} & \gamma_{LT} & & \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix} c\gamma_2 \begin{pmatrix} 1 \\ 0 \\ \beta'^2 \\ 0 \end{pmatrix}$$

$$= c\gamma_2 \begin{pmatrix} \gamma_{LT} \\ \gamma_{LT}\beta_{LT} \\ \beta'^2 \\ 0 \end{pmatrix} \underbrace{=}_{\text{Just taking out factor } \gamma_{LT}} c\gamma_2\gamma_{LT} \begin{pmatrix} 1 \\ \beta_{LT} \\ \beta'^2/\gamma_{LT} \\ 0 \end{pmatrix} \underbrace{=}_{\text{From definition of } u^\mu} c\gamma \begin{pmatrix} 1 \\ \beta^1 \\ \beta^2 \\ \beta^3 \end{pmatrix},$$

where γ arises from the total motion of particle in frame S.

From the zeroth element we can see that $\gamma = \gamma_2\gamma_{LT}$ and therefore $\beta^1 = \beta_{LT}$ and $\beta^2 = \beta'^2/\gamma_{LT}$, as obtained previously!

1(b) alternative answer following similar method

$$u'^{\mu} \doteq c\gamma_2 \begin{pmatrix} 1 \\ \beta'^1 \\ 0 \\ 0 \end{pmatrix}$$

$$u^{\mu} = \begin{pmatrix} \gamma_{LT} & \gamma_{LT}\beta_{LT} & 0 & 0 \\ \gamma_{LT}\beta_{LT} & \gamma_{LT} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \beta'^1 \\ 0 \\ 0 \end{pmatrix} c\gamma_2 = \begin{pmatrix} \gamma_{LT}(1+\beta_{LT}\beta'^1) \\ \gamma_{LT}(\beta_{LT}+\beta'^1) \\ 0 \\ 0 \end{pmatrix} c\gamma_2$$

$$= c\gamma \begin{pmatrix} 1 \\ \beta^1 \\ \beta^2 \\ \beta^3 \end{pmatrix}$$

$\therefore \quad \gamma = \gamma_2 \gamma_{LT}(1+\beta_{LT}\beta'^1)$ from zeroth element

and $\beta^1 = \dfrac{\gamma_2\gamma_{LT}(\beta_{LT}+\beta'^1)}{\gamma} = \dfrac{\beta_{LT}+\beta'^1}{1+\beta_{LT}\beta'^1}$,

as obtained previously

Q2(a)

$$\dot{\beta}^2 = \frac{d}{dt}\left(\beta^2\right) = \frac{\Delta\beta^2}{\Delta t} = \frac{c\,\Delta\beta^2}{\Delta x^0}$$

$$= \frac{c}{\gamma_{LT}\left(\Delta x'^0 + \underbrace{\beta_{LT}\Delta x'^1}_{=0}\right)} \underbrace{\left(\frac{\Delta\beta'^2}{\gamma_{LT}}\right)}_{\text{from Q1 (a)}}$$

$$= \frac{1}{\gamma_{LT}^2}\left[\frac{c\,\Delta\beta'^2}{\Delta x'^0}\right] = \frac{\dot{\beta}'^2}{\gamma_{LT}^2}$$

$$\therefore \boxed{\dot{\beta}^2 = \frac{\dot{\beta}'^2}{\gamma_{LT}^2}}$$

From Q1(a) the speed (and therefore the change of speed) is greater in S' by a factor γ_{LT}.

This change happens in a time interval that is shorter in S' by a factor γ_{LT}.

$\therefore$ The acceleration in S' is a factor γ_{LT}^2 greater than in S.

Q2(b) Using $\beta^1 = \dfrac{\beta_{LT} + \beta'^1}{1 + \beta_{LT}\beta'^1}$

$$\Delta\beta^1 = \frac{\Delta\beta'^1}{1+\beta_{LT}\beta'^1} + \frac{\beta_{LT}+\beta'^1}{\left(1+\beta_{LT}\beta'^1\right)^2}\cdot\left(-\beta_{LT}\Delta\beta'^1\right)$$

$$= \frac{\Delta\beta'^1}{\left(1+\beta_{LT}\beta'^1\right)^2}\left[1+\beta_{LT}\beta'^1 - \beta_{LT}^2 - \beta_{LT}\beta'^1\right]$$

$$\Delta\beta^1 = \frac{\Delta\beta'^1}{\gamma_{LT}^2\left(1+\beta_{LT}\beta'^1\right)^2}$$

$$\dot{\beta}^1 = c\frac{\Delta\beta^1}{\Delta x^0} = \frac{c\,\Delta\beta'^1}{\gamma_{LT}^2\left(1+\beta_{LT}\beta'^1\right)^2}\frac{1}{\gamma\left(\Delta x'^0 + \beta_{LT}\Delta x'^1\right)}$$

$$= \frac{c}{\gamma^3}\frac{\Delta\beta'^1}{\Delta x'^0}\frac{1}{\left(1+\beta_{LT}\beta'^1\right)^3}$$

$$\boxed{\dot{\beta}^1 = \frac{\dot{\beta}'^1}{\gamma_{LT}^3}\frac{1}{\left(1+\beta_{LT}\beta'^1\right)^3}}$$

We shall be interested in the case that, the particle is approximately at rest in S' so that $\dot{\beta}^1 \approx \dfrac{\dot{\beta}'^1}{\gamma_{LT}^3}$

In addition to the factor of $\frac{1}{\gamma_{LT}^2}$ we obtained in Q2 (a), which arises from the dilation of time intervals in S', when measured in S, we also have to consider that distances in S' along the direction of motion are contracted in S. This gives an extra factor of $\frac{1}{\gamma_{LT}}$.

Question 3

(a) Apply the divergence theorem to obtain flux of $\underline{B}$

$$\Phi_B = \oint \underline{B} \cdot \underline{da} = \int \nabla \cdot \underline{B} \, d\tau = \int \mu_0 g \, \delta^3(\underline{r}) \, d\tau$$

By symmetry $\underline{B}$ is radially outwards

$$\therefore \quad B \cdot 4\pi r^2 = \mu_0 g$$

$$\therefore \quad \underline{B} = \frac{\mu_0 g}{4\pi r^2} \, \hat{\underline{r}}$$

(b) For the given form of $\underline{A} = A_\phi \hat{\underline{\phi}}$ the only non-zero components of $\nabla \times \underline{A} = \underline{B}$ are as follows (in spherical polar coordinates)

$$\underline{B} = \nabla \times \underline{A} = \frac{\mu_0 g}{4\pi} \left\{ \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} \left(\sin\theta A_\phi \right) \right] \hat{\underline{r}} - \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r A_\phi \right) \right] \hat{\underline{\theta}} \right.$$

$$= \frac{\mu_0 g}{4\pi} \left\{ \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} \left(\frac{1 - \cos\theta}{r} \right) \right] \hat{\underline{r}} - \frac{1}{r} \left[\frac{\partial}{\partial r} \left(\frac{1 - \cos\theta}{\sin\theta} \right) \right] \hat{\underline{\theta}} \right.$$

$$= \frac{\mu_0 g}{4\pi} \left\{ \frac{1}{r^2 \sin\theta} \cdot \sin\theta \, \hat{\underline{r}} - 0 \right\}$$

$$= \frac{\mu_0 g}{4\pi r^2} \, \hat{\underline{r}} \quad \text{as obtained in (a) above.}$$