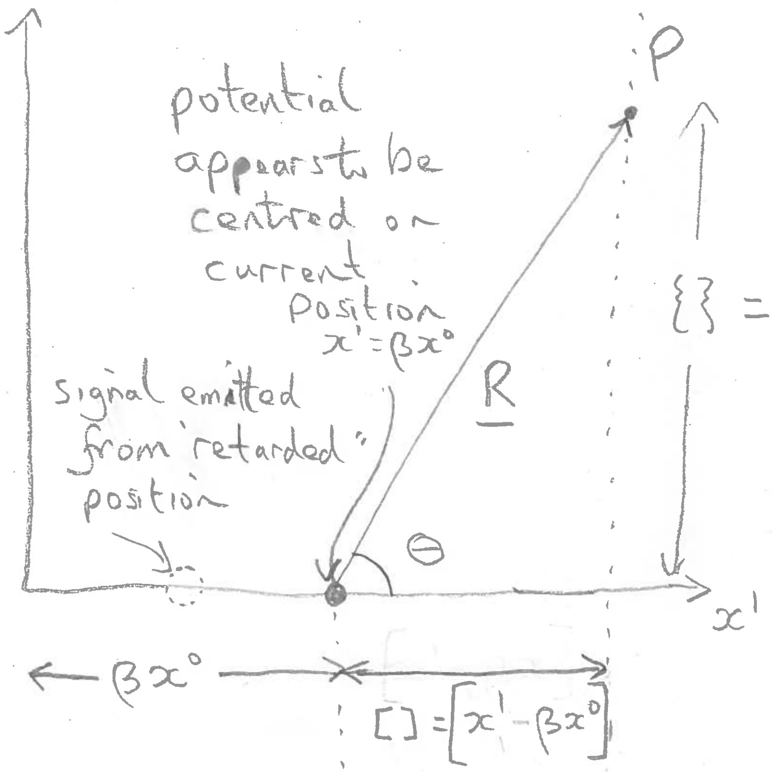
# PHYS30441

# Electrodynamics

# Example Sheet 3

Solutions — Terry Wyatt.

Q1) What does the potential look like in S!




$\{\} = \{(x^2)^2 + (x^3)^2\}^{1/2}$

$\sin\theta = \dfrac{\{\}}{R}$

Express $A^0$ in terms of $R = \left([x^1 - \beta x^0]^2 + (x^2)^2 + (x^3)^2\right)^{1/2}$

$= \left([\,]^2 + \{\}^2\right)^{1/2}$

the distance from "current" position of q and P

$$\frac{R'}{\gamma} = \frac{1}{\gamma}\left(\gamma^2[\ ]^2 + \{\ \}^2\right)^{1/2} = \left([\ ]^2 + (1-\beta^2)\{\ \}^2\right)^{1/2}$$

$$= \left(R^2 - \beta^2\{\ \}^2\right)^{1/2} = R\left(1 - \beta^2\frac{\{\ \}}{R^2}\right)^{1/2} = R\left(1-\beta^2\sin^2\theta\right)^{1/2}$$

$$A^0 = \frac{\gamma q}{4\pi\varepsilon_0 c R'} = \frac{q}{4\pi\varepsilon_0 c R}\,\frac{1}{\left(1-\beta^2\sin^2\theta\right)^{1/2}}$$

Q2) (a)

$$\underline{E} \cdot \underline{B} = E_1 B_1 + E_2 B_2 + E_3 B_3$$

From the transformation equations for E and B

$$E_1' B_1' = E_1 B_1$$

$$E_2' B_2' + E_3' B_3' = \gamma \left[ E_2 - \beta c B_3 \right] \gamma \left[ B_2 + \frac{\beta E_3}{c} \right]$$

$$+ \gamma \left[ E_3 + \beta c B_2 \right] \gamma \left[ B_3 - \frac{\beta E_2}{c} \right]$$

$$= \gamma^2 \left\{ \left[ 1 - \beta^2 \right] \left[ E_2 B_2 + E_3 B_3 \right] + \left[ \frac{\beta}{c} - \frac{\beta}{c} \right] E_2 E_3 + \left[ \beta c - \beta c \right] B_2 B_3 \right\}$$

$$= E_2 B_2 + E_3 B_3$$

$$\therefore \underline{E}' \cdot \underline{B}' = E_1' B_1' + E_2' B_2' + E_3' B_3' = \underline{E} \cdot \underline{B}$$

Q2(b)(i)

$$(E_2')^2 + (E_3')^2 - c^2\left[(B_2')^2 + (B_3')^2\right]$$

$$= \gamma^2\left\{(E_2 - \beta c B_3)^2 + (E_3 + \beta c B_2)^2 - c^2\left[\left(B_2 + \frac{\beta E_3}{c}\right)^2 + \left(B_3 - \frac{\beta E_2}{c}\right)^2\right]\right\}$$

$$= \gamma^2\left\{(E_2^2 + E_3^2)\left(1 - \frac{\cancel{c^2}\beta^2}{\cancel{c^2}}\right) - c^2(B_2^2 + B_3^2)(1 - \beta^2)\right\}$$

$$-2\gamma^2\left\{(E_2 B_3 - E_3 B_2)\left(\beta c - \frac{c^2\beta}{c}\right)\right\}$$

$$= E_2^2 + E_3^2 - c^2(B_2^2 + B_3^2)$$

Since also $E_1' = E_1$ and $E_2' = E_2$

$$\therefore (E')^2 - c^2(B')^2 = E^2 - c^2 B^2$$

Q2) (b) (ii) (optional challenge)

Firstly show that the elements of $F_{\mu\nu}$ are the same as those for $F^{\mu\nu}$ except for the replacement $E_i \to -E_i$

A) Using matrix multiplication

$$F_{\mu\nu} = g_{\mu\alpha} F^{\alpha\beta} g_{\beta\nu}$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix}$$

b) Alternative treatment using index notation

$$F_{0i} = g_{0\alpha} F^{\alpha\beta} g_{\beta i}$$

$$= \delta_{0\alpha} F^{\alpha\beta} (-\delta_{\beta i})$$

$$= -F^{0i}$$

$\therefore$ Signs of $E_i$ terms are reversed

$$F_{ij} = g_{i\alpha} F^{\alpha\beta} g_{\beta j}$$

$$= (-\delta_{i\alpha}) F^{\alpha\beta} (-\delta_{\beta j})$$

$$= F^{ij}$$

$\therefore$ Signs of $B_i$ terms are unchanged

$$
\begin{aligned}
F_{\mu\nu}F^{\mu\nu} = {} & F_{00}F^{00} + F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} \\
& + F_{10}F^{10} + F_{11}F^{11} + F_{12}F^{12} + F_{13}F^{13} \\
& + F_{20}F^{20} + F_{21}F^{21} + F_{22}F^{22} + F_{23}F^{23} \\
& + F_{30}F^{30} + F_{31}F^{31} + F_{32}F^{32} + F_{33}F^{33}
\end{aligned}
$$

$$
\begin{aligned}
= {} & 0 \quad - \frac{E_1^2}{c^2} \quad - \frac{E_2^2}{c^2} \quad - \frac{E_3^2}{c^2} \\
& - \frac{E_1^2}{c^2} \quad 0 \quad + B_3^2 \quad + B_2^2 \\
& - \frac{E_2^2}{c^2} \quad + B_3^2 \quad 0 \quad + B_1^2 \\
& - \frac{E_3^2}{c^2} \quad + B_2^2 \quad + B_1^2 \quad 0
\end{aligned}
$$

$$
= -\frac{2}{c^2}\left(E^2 - c^2 B^2\right)
$$

Since $F_{\mu\nu}F^{\mu\nu}$ must be a Lorentz scalar then so must $E^2 - c^2 B^2$ also.

Q2) (c)

Since in one frame of reference $B = 0$ at $P$

$E^2 - c^2 B^2$ must be positive (since $E \neq 0$ in that frame)

If in another frame $E = 0$ at $P$ then

$E^2 - c^2 B^2 \leq 0$ in that frame (since $-c^2 B^2 \leq 0$)

This would violate the fact that $E^2 - cB^2$ is a Lorentz invariant.