Extra-curricular reading

Local gauge symmetry
$$\underline{A} \to \underline{A} + \nabla \chi \qquad \chi \text{ varies "locally"}$$
$$V \to V - \frac{\partial \chi}{\partial t}$$

.... and its consequences for the Schroedinger Equation

Introducing gauge invariance†

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Abstract. A very important and beautiful feature of modern particle physics is use of the symmetry principle of local gauge invariance to predict/explain the existence of the force-carrying vector bosons: the photon, the gluon, the W and the Z. This is generally presented at a specialist level, using covariant derivatives and the relativistic Lagrangian. But it can also be explained in simpler terms, understandable by anyone with a slight knowledge of the Schrödinger equation. One such exposition is given here.

Résumé. Une caractéristique très importante et élégante de la physique des particules est l'utilisation de l'invariance de jauge locale pour prédire/expliquer l'existence des bosons vecteurs: le photon, le gluon, le W et le Z. On présente généralement cette theorie à un niveau spécialisé, en utilisant la dérivée covariante et le Lagrangien relativiste. On peut cependant l'expliquer dans une langage plus simple, compréhensible par quiconque connait l'équation de Schrödinger. C'est cette dernière approche qui est suivie ici.

In the beginning, as everyone knows, God created Heaven and Earth and quantum mechanics. But as he was admiring part of his handiwork

$$\Psi(x,t) = \exp[i(px - Et)/\hbar]$$

a less appreciative critic appeared on the scene.

'What's that, then?' asked the Devil.

'That,' explained God, 'is a wavefunction. It represents a free electron (or any other particle) with momentum p and energy E.'

'Oh yeah?' was the less-than-enthusiastic response. 'Says who?'

'Says Schrödinger.' came the crushing reply. 'In one dimension, with no potential energy (because this is a simple free particle), his equation runs

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\Psi = i\hbar\frac{d}{dt}\Psi$$

which for this Ψ function reduces (after a little differentiation) to

$$\frac{p^2}{2m}\Psi = E\Psi$$

which is trivially true, as $p^2/2m$ is just the kinetic energy, $\frac{1}{2}mv^2$.'

'That's all very nice,' said the Devil scornfully, 'but is it applicable? How does it relate to the real world of everyday experience?'

'Well,' said God, 'for one thing, it tells you the electron's probability density, which is very useful and practical thing to know.'

$$P(x,t) = \Psi^*(x,t)\Psi(x,t) = |\Psi(x,t)|^2.$$

'Humph.' said the Devil, unimpressed. 'It strikes me there's something odd about this. $P(x,t)$ only depends on the *magnitude* of Ψ. $-\Psi$ would give the same probability. So would any function $\Psi' = e^{ia}\Psi$. The phase of your 'wavefunction' is totally irrelevant and unnecessary. Seems a shaky basis for a universe.'

'No problem,' smiled the deity, 'there is an ambiguity, but it doesn't matter. If Ψ is a solution of the Schrödinger equation then so is $\Psi' = e^{ia}\Psi$; the phase factor is just a constant multiplying both sides of the equation, which remains true. Two functions differing only by a constant factor represent the same physical state. Does that satisfy you?'

'Not yet,' replied the Devil, with all the complacency of someone sliding an ace out of his sleeve, 'this multiplication by e^{ia} changes the phase of Ψ by some value a which is *globally* the same everywhere. This is alright for ubiquitous divinities like you and me, but this universe is supposed to run according to the laws of special relativity. That means that things happening simultaneously (in any reference frame) have to be independent, as messages can't get from one to the other instantly, but are limited to the speed of light. So the change of the phase has to be allowed to vary *locally*; to be different in different places and times

$$\Psi'(x,t) = \exp[i\alpha(x,t)]\,\Psi(x,t)$$

as there is no legal way of forcing it to be the same.'

† This is the introduction to a talk entitled 'Teaching the Standard Model' given at an Institute of Physics Meeting on 'Teaching High Energy Physics' at Imperial College, London, on 9th November 1988.

'Fair enough,' said God, 'it still makes no difference to $P(x, t)$.'

'No.' said the Devil, springing his trap. 'But look what it does to the Schrödinger Equation. $e^{i\alpha}$ is not just a constant: when you differentiate Ψ' with respect to t on the right-hand side of the equation you get a new term from the differential of x. This cannot be balanced by anything on the other side.

$$i\hbar\frac{d}{dt}\Psi' = i\hbar e^{i\alpha}\frac{d}{dt}\Psi - \hbar\frac{d\alpha}{dt}e^{i\alpha}\Psi.$$

On the left you get differentials with respect to α that can't be balanced in the right. So $\Psi'(x, t)$, which is supposed to represent the same state as the original $\Psi(x, t)$ as it differs only by a trivial phase factor, is *not* a solution of the Schrödinger equation after all. This shows that your creation is, as I'd always suspected, unsound, illogical, and fundamentally flawed.'

God thought about this for a few millenia before replying. 'Hmmm. You have got a point there. An artificial change to a wavefunction can't alter its nature. But special relativity and complex wavefunctions are fundamental to the way the universe works, and I can't do without them. So there must be something wrong—or, at least, incomplete—with the Schrödinger equation. I'll have to add a couple of new terms, one for each of the troublesome differentials, making

$$\frac{-\hbar^2}{2m}\left(\frac{d}{dx} - igA(x, t)\right)^2\Psi(x, t) - g\hbar\phi(x, t)\Psi(x, t)$$
$$= i\hbar\frac{d}{dt}\Psi(x, t)$$

where g is an arbitrary constant, and A and ϕ are functions for which the effect of α is *prescribed* as follows:

if $\quad \Psi(x, t) \to \exp[i\alpha(x, t)]\,\Psi(x, t)$

then $\quad \phi(x, t) \to \phi(x, t) + \frac{1}{g}\frac{d\alpha}{dt}$

and $\quad A(x, t) \to A(x, t) + \frac{1}{g}\frac{d\alpha}{dx}$.

'Now look what happens. If the phase of Ψ changes by α, the $g\hbar\phi\Psi$ term gives an extra $-(g\hbar/g)(d\alpha/dt)\Psi'$ on the left-hand side. This is exactly the same as your new term on the right, and the balance is restored. In the same way, the extra $d\alpha/dx$ term from the differentiation cancels exactly with the extra term from A, all within the big bracket. The equation balances again, and the universe does work after all.'

'OK.' sneered the Devil, after scribbling desperately on the back of an envelope and discovering that the cancellations happened just as described. 'So you've argued your way out of that one. But look what it's cost you—you started with one function, Ψ, and now you've got these extra A and ϕ functions as well. What are they supposed to be?'

'Interesting.' mused God. '$\Psi(x, t)$ already describes an electron, so A and ϕ must describe some other particle. I wonder what it is.'

'And another thing,' added the Devil, 'the real universe has three dimensions. So the differential in the Schrödinger equation has three components, and you need three separate functions to balance them.'

'Actually,' came the reply, 'that makes things clearer. Call them A_x, A_y and A_z. They are the three space components of a vector: this shows that the A particle must have a spin of one unit of angular momentum, as spin J particles have $(2J + 1)$ components. So it is a boson (jargon meaning 'integral spin particle'). The field ϕ is not something different, it's just the fourth member of the A 4-vector. We might as well have called it A_t.'

'It looks messy,' muttered the Devil, 'the simple Schrödinger equation has picked up all sorts of terms involving g, A and Ψ. What do they mean?'

'They must be some sort of energy,' said God, 'and as their values depend on the strengths of A and Ψ, it's an energy of interaction between particles: this A particle interacts with electrons. There are no terms like gAA, and furthermore any such terms would misbehave disasterously under one of these phase changes, so A does not interact with itself.'

'But this A particle must have a wave equation of its own,' objected the Devil in one last desparate attempt to win the argument, 'and that must remain valid when A changes because of α. How are you going to do that—is it going to need even more functions, and so on forever?'

'No, that's all straightforward,' God replied with more than a hint of satisfaction, 'there are some quantities like $(dA_x/dy) - (dA_y/dx)$ that don't change when α is introduced, and from them you can build (with a bit of manipulation: specifically, by applying the Lorentz condition) an equation

$$\nabla^2 A = \frac{1}{c^2}\frac{d^2 A}{dt^2}$$

which is a perfectly good wave equation (indeed, it's just the equation for simple harmonic motion). In terms of particles this is another energy equation; not $E = p^2/2m$ this time, but $E^2 = p^2c^2$, which is the (relativistic) energy equation for a particle of zero rest mass. So the A particle must be massless: there is no way that a mass term can be included if the invariance property is to be satisfied.'

'You mean to say,' said the Devil, 'that to keep the physical nature of the wavefunction the same under local changes of phase, you need a massless spin-1 boson, that interacts with electrons but not with itself?'

'Yes indeed.' replied God. 'I think I'll call it the photon. LET THERE BE LIGHT!'

And God saw that it was good. So good that He did it again using 2×2 matrices, and created the weak force bosons, and with 3×3 matrices for the strong force bosons, and maybe several other times as well to make supersymmetry and grand unified fields—but we are still trying to work out how.